Causality for Nonlocal Events

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1. Classical causal structure

In Lorentzian geometry, the causal structure of a spacetime $\mathcal{M}$ is modelled by a binary relation $\preceq$ on $\mathcal{M}$, with $p \preceq q$ meaning that the event p can be connected with the event q by means of a piecewise smooth future-directed causal curve (or $p = q$).

The relation $\preceq$ is a preorder. One calls a spacetime $\mathcal{M}$ *causally simple* if $\preceq$ is additionally *transitive* and *topologically closed* (as a subset of $\mathcal{M}^2$). In such case, the following 'dual' characterisation of $\preceq$ holds [1].

Theorem (Besnard, Minguzzi): Let $\mathcal{M}$ be a causally simple spacetime and let $\mathcal{C}(\mathcal{M})$ denote the set of all functions in $C_b^\infty(\mathcal{M})$ which are *causal*, i.e., non-decreasing along future-directed causal curves. Then for any $p, q \in \mathcal{M}$

$$p \preceq q \quad \iff \quad \forall\, f \in \mathcal{C}(\mathcal{M}) \quad f(p) \leq f(q).$$

2. Causal relation between probability measures

Drawing from the optimal transport theory adapted to the Lorentzian setting, in [1] we have proposed the following extension of the relation $\preceq$ onto $\mathscr{P}(\mathcal{M})$ – the space of Borel probability measures on $\mathcal{M}$.

Definition: Let $\mathcal{M}$ be a spacetime. For $\mu, \nu \in \mathscr{P}(\mathcal{M})$ we say that μ *causally precedes* ν (denoted $\mu \preceq \nu$) if there exists $\omega \in \mathscr{P}(\mathcal{M}^2)$ such that

(i) μ and ν are the left and right marginals of ω, respectively, (ii) $\omega(\preceq) = 1$.

The existence of such an ω, called a *causal coupling* of measures μ and ν, encapsulates the following intuitive notion of causality for 'spread' probability distributions:

Each infinitesimal portion of probability should travel along a worldline.

Theorem [1]: Let $\mathcal{M}$ be a causally simple spacetime. Then $\preceq$ is a partial order on $\mathcal{M}$ and

$$\begin{aligned} \mu \preceq \nu \quad &\iff \quad \forall\, f \in \mathcal{C}(\mathcal{M}) \ \int_{\mathcal{M}} f d\mu \leq \int_{\mathcal{M}} f d\nu \\ &\iff \quad \forall \text{ compact } \mathcal{K} \subseteq \operatorname{supp} \mu \quad \mu(\mathcal{K}) \leq \nu(J^+(\mathcal{K})), \end{aligned}$$

for any $\mu, \nu \in \mathscr{P}(\mathcal{M})$, where $J^+(\mathcal{K})$ denotes the causal future of the set $\mathcal{K}$.

3. Causal evolution of measures

Let $\mathcal{M}$ be a *globally hyperbolic* spacetime and let $\mathcal{T}$ be a fixed *Cauchy temporal function*, i.e., a function $\mathcal{T} \in C^\infty(\mathcal{M})$ whose level sets are spacelike Cauchy hypersurfaces. A curve $\gamma : I \to \mathcal{M}$ such that $\mathcal{T}(\gamma(t)) = t$ for any $t \in I$ is a worldline iff

$$\forall\, s, t \in I \quad s \leq t \implies \gamma(s) \preceq \gamma(t).$$

By analogy, we define a 'measure-valued worldline'.

Definition: A *causal evolution of a measure* is a map $\mu : I \to \mathscr{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\operatorname{supp} \mu_t \subseteq \mathcal{T}^{-1}(t)$ for all $t \in I$ and, moreover, satisfying

$$\forall s, t \in I \quad s \leq t \implies \mu_s \preceq \mu_t.$$

Example [2]: Let $\mathcal{M} = \mathbb{R}^{1+n}$ be the Minkowski spacetime and let $\mu_t := \delta_t \otimes \rho(t, .)\lambda$, where λ is the Lebesgue measure on $\mathbb{R}^n$ and $\rho : \mathcal{M} \to \mathbb{R}$ is assumed to satisfy the continuity equation $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$ with a subluminal velocity field ($\|\mathbf{v}\| \leq 1$). Then $t \mapsto \mu_t$ is a causal evolution of a measure.

The following theorem provides an alternative view on the causal evolution of measures, showing that this is in fact an observer-independent notion.

Theorem [3]: Consider a map $t \mapsto \mu_t \in \mathscr{P}(\mathcal{M})$ satisfying $\operatorname{supp} \mu_t \subseteq \mathcal{T}^{-1}(t)$ for all $t \in I$. Then this map is causal iff there exists a **probability measure σ on the space of worldlines** (endowed with a suitable topology) such that $\mu_t = (\mathrm{ev}_t)_\# \sigma$ for all $t \in I$.

4. Causality in noncommutative spacetimes

Observe that, by the Riesz representation theorem, $\mathscr{P}(\mathcal{M}) \simeq S(C_0(\mathcal{M}))$, i.e., the space of states on the C^*-algebra $C_0(\mathcal{M})$, with the pure states given by Dirac deltas δ_p for $p \in \mathcal{M}$. It is then natural to ask whether the causal structure can be consistently defined for **noncommutative spacetimes** – understood as the spaces of pure states of noncommutative algebras. This was achieved in [4] within the framework of *Lorentzian spectral triples* and motivated by the following results:

Example [4,5]: Let $\mathcal{M}$ be a globally hyperbolic spacetime with a spin structure $\mathcal{S}$. We obtain a (commutative) Lorentzian spectral triple $(\mathcal{A}, \mathcal{K}, \mathcal{D})$ by taking $\mathcal{A} := C_c^\infty(\mathcal{M})$, $\mathcal{K} := L^2(\mathcal{M}, \mathcal{S})$ equipped with an indefinite inner product $(\phi, \psi) = \int_{\mathcal{M}} \bar{\phi}\psi$, and $\mathcal{D} := \not{D} = -i\gamma^\mu \nabla_\mu^{\mathcal{S}}$, the Dirac operator associated with $\mathcal{S}$.

Theorem [4]: A function $a \in C_b^\infty(\mathcal{M})$ is causal iff $(\phi, [\not{D}, a]\phi) \leq 0$ for all $\phi \in \mathcal{K}$.

Definition [4]: For two states $\omega, \chi \in S(\mathcal{A})$ we define

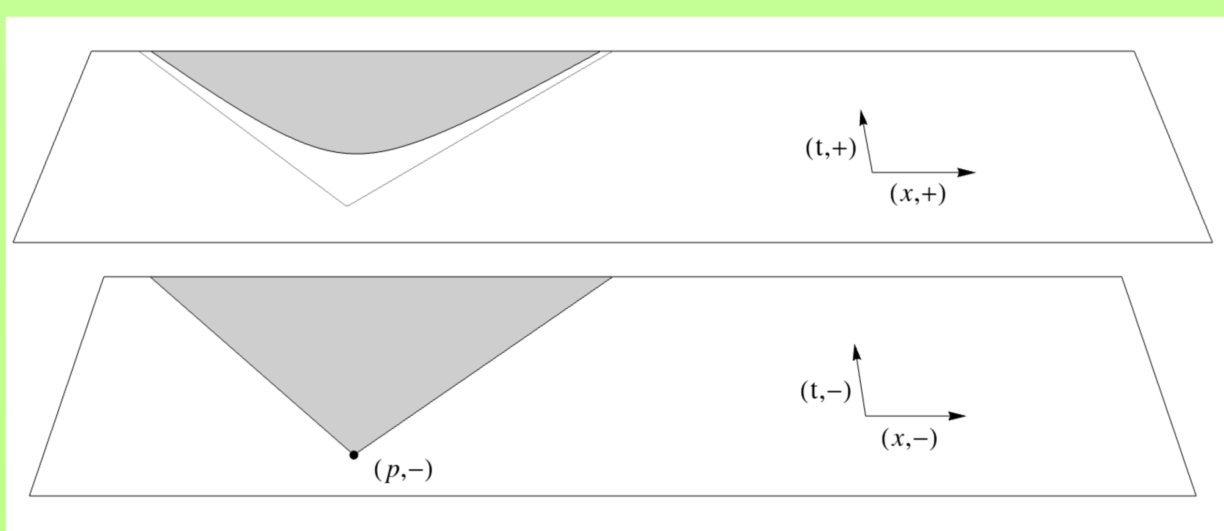
$$\omega \preceq \chi \quad \iff \quad \omega(a) \leq \chi(a) \quad \text{for all } a \in \mathcal{A} \text{, such that } \forall\, \phi \in \mathcal{K} \ \ (\phi, [\mathcal{D}, a]\phi) \leq 0.$$

Let us illustrate the above concepts with a simple *almost-commutative* example:

$$\mathcal{A} = C_c^\infty(\mathcal{M}) \otimes (\mathbb{C} \oplus \mathbb{C}), \quad \mathcal{K} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^2, \quad \mathcal{D} = \not{D} \otimes 1 + \gamma \otimes \left(\begin{smallmatrix} 0 & m \\ \bar{m} & 0 \end{smallmatrix}\right).$$

Although $\mathcal{A}$ is actually a commutative algebra and $P(\mathcal{A}) = \mathcal{M} \sqcup \mathcal{M} = \mathcal{M} \times \{\pm\}$, but the off-diagonal part of $\mathcal{D}$ allows one to "jump" between the sheets.

Theorem [5]: Let $\tau(\gamma)$ be the proper time along a causal curve γ in $\mathcal{M}$. Two states $(p, -), (q, +) \in P(\mathcal{A})$ are causally related with $(p, -) \preceq (q, +)$ iff there exists a causal curve γ giving $p \preceq q$ on $\mathcal{M}$ and such that $\tau(\gamma) \geq \pi/(2\,|m|)$.



Strikingly, the numerical value $\pi/(2\,|m|)$, which becomes $\pi\hbar/(2\,|m|\,c^2)$ once the physical dimensions are restored, is precisely the half-period of *Zitterbewegung* – the trembling motion of a massive Dirac fermion [5].

References

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