Pseudodifferential Operators on Noncommutative Tori

(joint work with Hyun-su Ha and Raphaël Ponge)

Gihyun Lee
Department of Mathematical Sciences,
Seoul National University,
Seoul, South Korea
gihyun.math@gmail.com

Abstract

Connes' pseudodifferential calculus on noncommutative n-tori [1] is a very powerful tool in noncommutative geometry. It allows us to derive heat kernel expansions of geometric operators on noncommutative tori, and geometric information such as curvature can be decoded from this expansion (see, e.g., [2, 3, 4, 6, 7]). However, detailed proofs of the main results of Connes' pseudodifferential calculus have still not appeared anywhere. The aim of this poster is to present an approach to fill this gap. We also announce some results on traces on pseudodifferential operators on noncommutative tori. This is joint work with Hyun-su Ha and Raphaël Ponge.

1 Noncommutative Tori

Let $\theta \in M_n(\mathbb{R})$, $n \geq 2$ be a skew-symmetric matrix. The *noncommutative torus* $\mathcal{A}_\theta$ arises from a deformation of the product on $C^\infty(\mathbb{T}^n)$. It is a Fréchet $*$-algebra generated by n unitaries $U_1, \ldots, U_n$ such that

$$U_j U_k = e^{2\pi i \theta_{kj}} U_k U_j.$$

In what follows for $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ denote by U^k the element $U_1^{k_1} \cdots U_n^{k_n}$ of $\mathcal{A}_\theta$. Any $u \in \mathcal{A}_\theta$ can be written in the form,

$$u = \sum_{k \in \mathbb{Z}^n} u_k U^k, \qquad (u_k)_{k\in\mathbb{Z}^n} \in \mathcal{S}(\mathbb{Z}^n),$$

where $\mathcal{S}(\mathbb{Z}^n)$ is the space of rapidly decaying sequences.

The noncommutative torus $\mathcal{A}_\theta$ carries an action of $\mathbb{R}^n$ determined by $\alpha_s(U^k) = e^{is\cdot k} U^k$, $s \in \mathbb{R}^n$, $k \in \mathbb{Z}^n$. This action gives rise to derivations $\delta_1, \ldots, \delta_n : \mathcal{A}_\theta \to \mathcal{A}_\theta$ given by

$$\delta_j(U_k) = \delta_{jk} U_k, \qquad j, k = 1, \ldots, n.$$

There is a unique normalized trace τ on $\mathcal{A}_\theta$ given by $\tau(\sum_{k\in\mathbb{Z}^n} u_k U^k) = u_0$. We denote by $\mathcal{H}$ the Hilbert space obtained as the completion of $\mathcal{A}_\theta$ with respect to the inner product $\langle u, v\rangle_{\mathcal{H}} := \tau(v^* u)$, $u, v \in \mathcal{A}_\theta$.

Let $\|\cdot\|$ be the norm on $\mathcal{A}_\theta$ defined by $\|u\| := \sup_{\|v\|_{\mathcal{H}}=1} \|uv\|_{\mathcal{H}}$. Then we endow $\mathcal{A}_\theta$ with the Fréchet-Montel space topology generated by the seminorms $u \to \|\delta^\alpha u\|$, where $\delta^\alpha := \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ ranges over $\mathbb{N}_0^n$.

2 Amplitudes and Oscillating Integrals

Definition 2.1. *$A^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$, $m \in \mathbb{R}$, consists of maps $a(s,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$ such that, for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, there is $C_{\alpha\beta\gamma} > 0$ such that*

$$(2.1)\quad \|\delta^\alpha \partial_s^\beta \partial_\xi^\gamma a(s,\xi)\| \leq C_{\alpha\beta\gamma}(1+|s|+|\xi|)^m \quad \forall (s,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

We endow $A^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$ with the Fréchet space topology generated by the seminorms given by the best constants $C_{\alpha\beta\gamma}$ in the estimates (2.1).

For $a(s,\xi) \in A^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$ with $m < -2n$, define $J_0(a) := \iint e^{is\cdot\xi} a(s,\xi) ds đ\xi$. Let $\chi \in C_c(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $\chi = 1$ near the origin and set $L = \chi(s,\xi) + \frac{1-\chi(s,\xi)}{|s|^2+|\xi|^2} \sum_{j=1}^n (\xi_j D_{s_j} + s_j D_{\xi_j})$. Then we have $L(e^{is\cdot\xi}) = e^{is\cdot\xi}$. Moreover, for every $m \in \mathbb{R}$, the transpose operator L^t gives rise to a continuous linear map from $A^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$ to $A^{m-1}(\mathbb{R}^n \times \mathbb{R}^n)$.

Proposition 2.2 ([8]). *There is a unique linear map $a \to J(a)$ from $A^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta) := \cup_{m\in\mathbb{R}} A^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$ to $\mathcal{A}_\theta$ such that*

(1) It restricts to a continuous map on $A^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$ for all $m \in \mathbb{R}$.

(2) For all $a(s,\xi) \in \cup_{m<-2n} A^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$, we have

$$J(a) = J_0(a) = \iint e^{is\cdot\xi} a(s,\xi) ds đ\xi.$$

(3) If $a(s,\xi) \in A^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$, $m \in \mathbb{R}$, we have

$$J(a) = \iint e^{is\cdot\xi} (L^t)^N a(s,\xi) ds đ\xi,$$

where N is any integer $N > m + 2n$.

3 Symbols and ΨDOs on $\mathcal{A}_\theta$

Definition 3.1. *(1) $\mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta)$, $m \in \mathbb{R}$, consists of maps $\rho(\xi) \in C^\infty(\mathbb{R}^n; \mathcal{A}_\theta)$ such that, for all $\alpha, \beta \in \mathbb{N}_0^n$, there exists $C_{\alpha\beta} > 0$ such that*

$$(3.1)\quad \|\delta^\alpha \partial_\xi^\beta \rho(\xi)\| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|} \quad \forall \xi \in \mathbb{R}^n.$$

(2) $\mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$, $m \in \mathbb{R}$, consists of maps $\sigma(s,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$ such that, for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$, there exists $C_{\alpha\beta\gamma} > 0$ such that

$$(3.2)\quad \|\delta^\alpha \partial_s^\beta \partial_\xi^\gamma \sigma(s,\xi)\| \leq C_{\alpha\beta\gamma}(1+|\xi|)^{m-|\gamma|} \quad \forall (s,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

We endow $\mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta)$ (resp., $\mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$) with the Fréchet space topology defined by the seminorms given by the best constants $C_{\alpha\beta}$ (resp., $C_{\alpha\beta\gamma}$) in the estimates (3.1) (resp., (3.2)).

Definition 3.2. *$S_m(\mathbb{R}^n; \mathcal{A}_\theta)$, $m \in \mathbb{C}$, consists of smooth maps $\rho : \mathbb{R}^n \backslash 0 \to \mathcal{A}_\theta$ such that $\rho(\lambda\xi) = \lambda^m \rho(\xi)$ for all $\xi \in \mathbb{R}^n \backslash 0$ and $\lambda > 0$.*

Definition 3.3. *$S^m(\mathbb{R}^n; \mathcal{A}_\theta)$, $m \in \mathbb{C}$, consists of maps $\rho(\xi) \in C^\infty(\mathbb{R}^n; \mathcal{A}_\theta)$ that admit an asymptotic expansion $\rho(\xi) \sim \sum_{j\geq 0} \rho_{m-j}(\xi)$, $\rho_{m-j}(\xi) \in S_{m-j}(\mathbb{R}^n; \mathcal{A}_\theta)$ in the sense that, for all $N \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{N}_0^n$, there exists $C_{N\alpha\beta} > 0$ such that, for all $\xi \in \mathbb{R}^n$, $|\xi| \geq 1$, we have*

$$\|\delta^\alpha \partial_\xi^\beta (\rho - \sum_{j<N} \rho_{m-j})(\xi)\| \leq C_{N\alpha\beta} |\xi|^{\Re m - N - |\beta|}.$$

Let $m \in \mathbb{R}$. As it turns out (*cf.* [8]), the map $(\rho, u) \to \rho(\xi)\alpha_{-s}(u)$ gives rise to a continuous bilinear map from $\mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta) \times \mathcal{A}_\theta$ to $A^{m^+}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$, where $m^+ := \max\{m, 0\}$. Let $\rho(\xi) \in \mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta)$. We define a ΨDO $P_\rho : \mathcal{A}_\theta \to \mathcal{A}_\theta$ by letting

$$P_\rho(u) := J(\rho(\xi)\alpha_{-s}(u)), \qquad u \in \mathcal{A}_\theta.$$

Using Proposition 2.2 we obtain the following result.

Proposition 3.4 ([8]). *Let $m \in \mathbb{R}$. Then the map $(\rho, u) \to P_\rho(u)$ gives rise to a continuous bilinear map from $\mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta) \times \mathcal{A}_\theta$ to $\mathcal{A}_\theta$.*

Let $m \in \mathbb{C}$. We denote by $\Psi^m(\mathcal{A}_\theta)$ the space of ΨDOs associated with classical symbols of order m and we denote by $\Psi^{-\infty}(\mathcal{A}_\theta)$ the space of ΨDOs associated with symbols in $\mathbb{S}^{-\infty}(\mathbb{R}^n; \mathcal{A}_\theta) := \cap_{m\in\mathbb{R}} \mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta)$.

4 Composition and Adjoints of ΨDOs

Given $m_1, m_2 \in \mathbb{R}$, let $\sigma(s,\xi) \in \mathbb{S}^{m_1}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$ and $\rho(\xi) \in \mathbb{S}^{m_2}(\mathbb{R}^n; \mathcal{A}_\theta)$. Consider the smooth map $\sigma\natural\rho : (\mathbb{R}^n)^3 \to \mathcal{A}_\theta$ defined by $\sigma\natural\rho(\xi; t, \eta) := \sigma(t, \xi+\eta)\alpha_{-t}[\rho(\xi)]$, $(\xi, t, \eta) \in (\mathbb{R}^n)^3$. It can be shown that $\sigma\natural\rho(\xi; \cdot, \cdot)$ belongs to $A^{|m_1|+|m_2|}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$ for all $\xi \in \mathbb{R}^n$. We then form the map $\sigma\#\rho$ from $\mathbb{R}^n$ to $\mathcal{A}_\theta$ defined by $\sigma\#\rho(\xi) := J(\sigma\natural\rho(\xi; \cdot, \cdot))$, $\xi \in \mathbb{R}^n$.

As it turns out (*cf.* [8]), the map $(\sigma, \rho) \to \sigma\#\rho$ gives rise to a continuous bilinear map from $\mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta) \times \mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta)$ to $\mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta)$. Since there is a continuous inclusion $\mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta) \hookrightarrow \mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$, we see that $\#$ induces a continuous bilinear map from $\mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta) \times \mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta)$ to $\mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta)$.

Theorem 4.1 ([8]). *Let $P \in \Psi^{m_1}(\mathcal{A}_\theta)$ and $Q \in \Psi^{m_2}(\mathcal{A}_\theta)$. Then $PQ \in \Psi^{m_1+m_2}(\mathcal{A}_\theta)$. Moreover, if P has symbol $\rho(\xi) \sim \sum_{j\geq 0} \rho_{m_1-j}(\xi)$ and Q has symbol $\sigma(\xi) \sim \sum_{j\geq 0} \sigma_{m_2-j}(\xi)$, then PQ has symbol $\rho\#\sigma(\xi)$ and we have*

$$\rho\#\sigma(\xi) \sim \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha \rho(\xi) \delta^\alpha \sigma(\xi).$$

In particular, PQ has principal symbol $\rho_{m_1}(\xi)\sigma_{m_2}(\xi)$.

Let $\rho(\xi) \in \mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta)$, $m \in \mathbb{R}$, and consider the double symbol $\rho^\dagger(s,\xi) \in \mathbb{S}^m(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{A}_\theta)$ given by $\rho^\dagger(s,\xi) := \alpha_{-s}(\rho(\xi)^*)$. We denote by $\rho^\star(\xi)$ the symbol in $\mathbb{S}^m(\mathbb{R}^n; \mathcal{A}_\theta)$ defined by $\rho^\star(\xi) := \rho^\dagger\#1(\xi)$.

Theorem 4.2 ([8]). *Let $P \in \Psi^m(\mathcal{A}_\theta)$. Then the formal adjoint P^* belongs to $\Psi^{\overline{m}}(\mathcal{A}_\theta)$. Moreover, if P has symbol $\rho(\xi) \sim \sum_{j\geq 0} \rho_{m-j}(\xi)$, then P^* has symbol $\rho^\star(\xi)$ and we have*

$$\rho^\star(\xi) \sim \sum_\alpha \frac{1}{\alpha!} \delta^\alpha \partial_\xi^\alpha \rho(\xi)^*.$$

In particular, P^* has principal symbol $\rho_m(\xi)^*$.

5 Ellipticity and Parametrices

Definition 5.1. *Let $P \in \Psi^m(\mathcal{A}_\theta)$ have symbol $\rho(\xi) \sim \sum_{j\geq 0} \rho_{m-j}(\xi)$. Then P is elliptic when $\rho_m(\xi)$ is invertible in $\mathcal{A}_\theta$ for all $\xi \in \mathbb{R}^n \backslash 0$.*

Remark 5.2. If $\rho_m(\xi)$ is invertible for all $\xi \in \mathbb{R}^n \backslash 0$, then $\rho_m(\xi)^{-1} \in C^\infty(\mathbb{R}^n \backslash 0; \mathcal{A}_\theta)$.

Proposition 5.3 ([8]). *An operator $P \in \Psi^m(\mathcal{A}_\theta)$ is elliptic if and only if there is $Q \in \Psi^{-m}(\mathcal{A}_\theta)$ such that $PQ = QP = 1$ modulo $\Psi^{-\infty}(\mathcal{A}_\theta)$.*

6 L^2-Boundedness and Action on Sobolev Spaces of ΨDOs

In what follows we endow the topological dual $\mathcal{A}_\theta{}'$ with its strong topology.

Proposition 6.1 ([8]). *Let $P \in \Psi^m(\mathcal{A}_\theta)$. Then P uniquely extends to a continuous linear map from $\mathcal{A}_\theta{}'$ to itself.*

Let $\Delta := \delta_1^2 + \cdots + \delta_n^2$ be the (flat) Laplacian on $\mathcal{A}_\theta$. Then, for all $s \in \mathbb{R}$, the operator $\Lambda^s := (1+\Delta)^{\frac{s}{2}}$ is in $\Psi^s(\mathcal{A}_\theta)$ and has symbol $\langle\xi\rangle^s$.

Definition 6.2. *The Sobolev space $\mathcal{H}^s$, $s \in \mathbb{R}$, consists of all $u \in \mathcal{A}_\theta{}'$ such that $\Lambda^s u \in \mathcal{H}$.*

The space $\mathcal{H}^s$ becomes a Hilbert space once equipped with the inner product, $\langle u, v\rangle_{\mathcal{H}^s} := \langle \Lambda^s u, \Lambda^s v\rangle_{\mathcal{H}}$, $u, v \in \mathcal{H}^s$. For $s' \geq s$ there is a continuous inclusion of $\mathcal{H}^{s'}$ in $\mathcal{H}^s$. Moreover, if $s' > s$ then the inclusion map $\mathcal{H}^{s'} \hookrightarrow \mathcal{H}^s$ is a compact operator.

Proposition 6.3 ([8]). *Let $P \in \Psi^m(\mathcal{A}_\theta)$, and set $a = \Re m$. Then*

(1) P induces a continuous linear operator from $\mathcal{H}^s$ to $\mathcal{H}^{s-a}$ for every $s \in \mathbb{R}$.

(2) If $a \leq 0$, then P induces a bounded operator of $\mathcal{L}(\mathcal{H})$. This operator is compact when $a < 0$.

Proposition 6.4 ([8]). *Let $P \in \Psi^m(\mathcal{A}_\theta)$ be elliptic and set $a = \Re m$. Given any $u \in \mathcal{A}_\theta{}'$, if $Pu \in \mathcal{H}^s$, then $u \in \mathcal{H}^{s+a}$.*

7 Schatten Classes and Trace Formulas

For $p > 0$ the Schatten class $\mathcal{L}^p$ consists of operators $T \in \mathcal{L}(\mathcal{H})$ such that $\operatorname{Tr}|T|^p < \infty$.

Proposition 7.1. *Let $P \in \Psi^m(\mathcal{A}_\theta)$, $\Re m < 0$, and set $p = \frac{|\Re m|}{n}$. Then*

(1) P is in the Schatten class $\mathcal{L}^q$ for every $q > p$.

(2) If $\Re m < -n$ and P has symbol $\rho(\xi)$, then

$$\operatorname{Tr}(P) = \tau\Big(\sum_{k\in\mathbb{Z}^n} \rho(k)\Big).$$

The weak Schatten class $\mathcal{L}^{p+}$ consists of compact operators T such that $\mu_k(T) = \mathrm{O}(k^{-\frac{1}{p}})$ as $k \to \infty$, where $\mu_k(T)$ is the $(k+1)$-th eigenvalue of $|T|$ counted with multiplicity (*cf.* Sukochev's lectures).

Theorem 7.2 ([8, 10]). *Let $P \in \Psi^m(\mathcal{A}_\theta)$, $\Re m < 0$, and set $p = \frac{|\Re m|}{n}$.*

(1) P is in the weak Schatten class $\mathcal{L}^{p+}$.

(2) If $m = -n$, then, for every (continuous) normalized trace $\mathcal{T}$ on $\mathcal{L}^{1+}$, we have

$$(7.1)\qquad \mathcal{T}(P) = \frac{1}{n}(2\pi)^{-n} \int_{|\xi|=1} \tau(\rho_{-n}(\xi)) d^{n-1}\xi,$$

where $\rho_{-n}(\xi)$ is the principal symbol of P.

Remark 7.3. When $n = 2$ Fathizadeh-Khalkhali [5] proved the trace formula (7.1) in the special case of Dixmier traces.

8 NC Residue and Traces on ΨDOs

Set $\Psi^{\mathbb{Z}}(\mathcal{A}_\theta) := \cup_{m\in\mathbb{Z}} \Psi^m(\mathcal{A}_\theta)$ and $S^{\mathbb{Z}}(\mathbb{R}^n; \mathcal{A}_\theta) := \cup_{m\in\mathbb{Z}} S^m(\mathbb{R}^n; \mathcal{A}_\theta)$. Given any symbol $\rho(\xi) \in S^{\mathbb{Z}}(\mathbb{R}^n; \mathcal{A}_\theta)$, we define the noncommutative residue of P_ρ by

$$\operatorname{Res} P_\rho = \int_{|\xi|=1} \tau(\rho_{-n}(\xi)) d^{n-1}\xi,$$

where $\rho_{-n}(\xi)$ is the symbol of degree $-n$ of $\rho(\xi)$. We also set $\Psi^{\mathrm{int}}(\mathcal{A}_\theta) := \cup_{\Re m<-n} \Psi^m(\mathcal{A}_\theta)$ and $\Psi^{\mathbb{C}\mathbb{Z}}(\mathcal{A}_\theta) := \cup_{m\in\mathbb{C}\backslash\mathbb{Z}} \Psi^m(\mathcal{A}_\theta)$.

We can define holomorphic families of symbols and ΨDOs on $\mathcal{A}_\theta$ much like as in the commutative case.

Theorem 8.1 ([9]). *(1) The standard trace $\Psi^{\mathrm{int}}(\mathcal{A}_\theta) \ni P \to \operatorname{Tr}(P)$ has a unique analytic extension $P \to \operatorname{TR}(P)$ to $\Psi^{\mathbb{C}\mathbb{Z}}(\mathcal{A}_\theta)$.*

(2) Let $P \in \Psi^{\mathbb{Z}}(\mathcal{A}_\theta)$ and $(P(z))_{z\in\mathbb{C}}$ a holomorphic family of ΨDOs such that $P(0) = P$ and $\operatorname{ord}P(z) = z + \operatorname{ord}P$. Then $z \to \operatorname{TR}[P(z)]$ has at worst a simple pole singularity near $z = 0$ such that

$$\operatorname{Res}_{z=0} \operatorname{TR}[P(z)] = -\operatorname{Res} P.$$

(3) TR and Res are trace functionals on $\Psi^{\mathbb{C}\mathbb{Z}}(\mathcal{A}_\theta)$ and $\Psi^{\mathbb{Z}}(\mathcal{A}_\theta)$, respectively.

Remark 8.2. In (2) we may take $P(z) = P(1+\Delta)^{\frac{z}{2}}$.

Proposition 8.3 ([10]). *Every trace functional on $\Psi^{\mathbb{C}\mathbb{Z}}(\mathcal{A}_\theta)$ (resp., $\Psi^{\mathbb{Z}}(\mathcal{A}_\theta)$) is a constant multiple of TR (resp., Res).*

References

[1] Connes, A.: *C*-algèbres et géométrie differentielle*. C.R. Acad. Sc. Paris, 290 (1980), Série A, 599–604.

[2] Connes, A.; Fathizadeh, F.: *The term a_4 in the heat kernel expansion of noncommutative tori*. E-print, arXiv:1611.09815, 134 pages.

[3] Connes, A.; Moscovici, H.: *Modular curvature for noncommutative two-tori*. J. Amer. Math. Soc. 27 (2014), 639–684.

[4] Connes, A.; Tretkoff, P.: *The Gauss-Bonnet theorem for the noncommutative two torus*. Johns Hopkins Univ. Press, 2011. pp. 141–158

[5] Fathizadeh, F.; Khalkhali, M.: *Weyl's law and Connes' trace theorem for noncommutative two tori*. Lett. in Math. Phys. 103 (2013), 1–18.

[6] Fathizadeh, F.; Khalkhali, M.: *Scalar curvature for noncommutative four-tori*. J. Noncommut. Geom. 9 (2015), 473–503.

[7] Floricel, R.; Ghorbanpour, A.; Khalkhali, M.: *The Ricci curvature in noncommutative geometry*. E-print, arXiv:1612.06688, 26 pages.

[8] Ha, H.; Lee, G.; Ponge, R.: *Pseudodifferential calculus on noncommutative tori*. In preparation.

[9] Ha, H.; Lee, G.; Ponge, R.: *Noncommutative residue for pseudodifferential operators on noncommutative tori*. In preparation.

[10] Ponge, R.: *Personal communication*.