Noncommutative Gauge Theory

Patrizia Vitale

Dipartimento di Fisica Università di Napoli Federico II and INFN

Noncommutative Geometry and Applications to Quantum Physics
Quy-Nhon, Vietnam 13.7-22.7 2017

Outline

Lecture III

- Motivation
- Symplectic realization of star products
- The NC algebra $\mathbb{R}^3_\lambda$
- The matrix basis for $\mathbb{R}^3_\lambda$
- The differential calculus
- NC gauge models on $\mathbb{R}^3_\lambda$
- Perspectives

Motivation

We have seen in previous lectures that, up to now, we have not succeeded in defining gauge models based on Moyal noncommutativity which are renormalizable.

The simplest generalization of constant noncommutativity is to consider linear noncommutativity

$$[x^i, x^j]_\star = c^{ij}_k x^k \quad i, j, k = 1, ., 3$$

The idea: In order to get new star products with star commutators of coordinate functions reproducing the commutator above, we investigate the possibility of identifying Moyal subalgebras using a kind of Jordan-Schwinger map.

Linear Poisson brackets in 3d and the JS map

What is the Jordan-Schwinger map

- Quantum: Hermitian realization of the angular momentum operator algebra ($su(2)$) in terms of quadratic polynomials of $\hat{a}_b^\dagger, \hat{a}_b, b = 1, 2$ (quantum JS map) [Schwinger65]
- Classical: symplectic realization of linear 3-d Poisson algebras with functions belonging to the canonical Poisson algebra on $\mathbb{R}^4$ (classical JS map) [MMVZ93]

The goal is to realize all 3-d Lie algebras as subalgebras of the Moyal algebra, each one with a different induced star product.

Linear Poisson brackets in 3d and the JS map

- Consider the algebra $\mathfrak{isp}(4,\mathbb{R})$. It can be realized as a Poisson algebra in $(\mathbb{R}^4, \{y_i, y_j\} = E_{ij})$

$$with \qquad E = \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix} \qquad and \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  using quadratic-linear functions on $\mathbb{R}^4$

- Homogeneous sector $\mathfrak{sp}(4,\mathbb{R})$:
  $M \in \mathbb{M}_4(\mathbb{R}) : M^T J + JM = 0, [M_a, M_b] = C_{ab}^c M_c, a = 1, ..10$
  To the symmetric matrices $B_a = -JM_a$ are associated quadratic functions $w_a = \frac{1}{2} y^T B_a y$

$$\{w_a, w_b\} = K_{ab}^c w_c$$

- linear functions y_i yield the inhomogeneous sector (translations).
- Why $\mathfrak{isp}(4,\mathbb{R})$? Because all three dimensional Lie algebras are Lie subalgebras of $\mathfrak{isp}(4,\mathbb{R})$

Linear Poisson brackets in 3d and the JS map

- A Poisson realization can be given on identifying the generators x_i with linear functions on the dual $\mathfrak{g}^* = \mathbb{R}^3$

  $$\{x_1, x_2\} = cx_3 + hx_2; \quad \{x_2, x_3\} = ax_1; \quad \{x_3, x_1\} = bx_2 - hx_3$$

  Jacobi identity is equivalent to $ah = 0$
- type A algebras h$= 0$ ($\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$ and contractions, down to the Abelian algebra)
- type B algebras a$= 0$ ($\mathfrak{sb}(2,\mathbb{C})$ and generalizations, no Casimir function)
- They correspond to subalgebras of $\mathfrak{isp}(4,\mathbb{R})$ and are realized as Poisson algebras in terms of quadratic-linear functions on $\mathbb{R}^4$
- The map $\pi : \mathbb{R}^4 \to \mathbb{R}^3 : \pi^* x_i = w_i(y^\mu)$ is the classical JS map

Linear Poisson brackets in 3d and the JS map

Examples:

- $\mathfrak{u}(2), \mathfrak{u}(1,1)$

$$\begin{aligned}\pi^* x_i &= \frac{1}{2}\bar{z}_a e_i^{ab} z_b, \quad z_1 = y_1 + iy_2,\ z_2 = y_3 + iy_4 \\ \pi^* x_0 &= \frac{1}{2}\bar{z}_a e_0^{ab} z_b \ \text{ Casimir function} \\ \{\pi^* x_i, \pi^* x_j\} &= c_{ij}^k x_k; \quad \{\pi^* x_0, \pi^* x_j\} = 0\end{aligned}$$

- $\mathfrak{sb}(2,\mathbb{C})$ (κ-Minkowski in $3-d$)

$$\begin{aligned}\pi^* x_1 &= -(y_1 y_2 + y_3 y_4), \quad \pi^* x_2 = y_1, \quad \pi^* x_3 = y_3 \\ \{\pi^* x_1, \pi^* x_{2,3}\} &= \pi^* x_{2,3} \quad \{\pi^* x_2, \pi^* x_3\} = 0\end{aligned}$$

These are symplectic realizations of Lie algebras

Quantum analogue: Hermitian realizations with $z_b \to \hat{a}_b$, $\bar{z}_b \to \hat{a}_b^\dagger$

Moyal subalgebras

Theorem [Gracia-Bondia Lizzi Marmo Vitale '02]: Each algebra of functions on $\mathbb{R}^3$ identified by the reduction map π is closed with respect to the Moyal $\star$ product.
New $\star$ products are induced, $\star_{\mathfrak{g}}$ defined by

$$\pi^*(f \star_{\mathfrak{g}} g) = \pi^* f \star_\theta \pi^* g$$

Intermediate step: Using the isomorphism between $\mathfrak{isp}(4, \mathbb{R})$ and quadratic linear functions on $\mathbb{R}^4$ $(b, M) \mapsto w_{(b,B)} = \frac{1}{2} y^T B y + b^T y$
B=JM

$$w \star_\theta f(y) = wf(y) + \frac{i\theta}{2}(By + b)^T J \,\mathrm{grad}\, f(y) + \frac{\theta^2}{8} \mathrm{Tr}\,[BJ \mathrm{Hess} f(y) J]$$

which yields, for the $\mathfrak{su}(2)$ case

$$x_j \star_{\mathfrak{su}(2)} f(x) = \{x_j - i\frac{\theta}{2}\varepsilon_{jlm} x_l \partial_m - \frac{\theta^2}{8}[(1 + x \cdot \partial)\partial_j - \frac{1}{2} x_j \Delta]\} f(x)$$

The Wick product

Remark:

The same reduction can be applied to a variation of the Moyal product, the Wick-Voros product (normal ordered)

$$\phi \star_W \psi(\bar{z}_a, z_a) = \exp\left[\theta \frac{\partial}{\partial \bar{z}_a} \frac{\partial}{\partial \bar{u}_a}\right] f(\bar{z}, z) g(\bar{u}, u)$$

The advantage of this product is that, when reduced to $\mathbb{R}^3$ it can be written in closed form as [Hammou Lagraa Sheikh-Jabbari '02]

$$\phi \tilde{\star} \psi\,(x) = \exp\left[\frac{\lambda}{2}\left(\delta^{ij} x_0 + i \epsilon^{ij}_{k} x^k\right) \frac{\partial}{\partial w^i} \frac{\partial}{\partial v^j}\right] \phi(w) \psi(v)|_{w=v=x}$$

The two products agree at the level of star-commutators and are related by an invertible map T [Kontsewich, Pinzul-Stern]

$$T(\phi \star \psi) = T\phi \tilde{\star} T\psi$$

The algebra R^3_λ

The NC algebra $(\mathcal{F}(\mathbb{R}^3), \star_{\mathfrak{su}(2)})$ is known as $\mathbb{R}^3_\lambda$.

- The star product

$$\phi \star \psi\,(x) = \exp\left[\frac{\lambda}{2}\left(\delta^{ij}x_0 + i\epsilon^{ij}_k x^k\right)\frac{\partial}{\partial u^i}\frac{\partial}{\partial v^j}\right]\phi(u)\psi(v)|_{u=v=x}$$

  implies, for coordinate functions

$$\begin{aligned} [x^i, x^j]_\star &= i\lambda\epsilon^{ij}_k x^k, \quad i = 1,..,3 \\ [x^0, x^j]_\star &= 0 \\ x^0 \star x^0 &= x^0(x^0 + \frac{\lambda}{2}) = \sum_i x^i \star x^i - \lambda x^0 \end{aligned}$$

- x^0 $\star$-commutes with x^i so that we can alternatively define $\mathbb{R}^3_\lambda$ as the $\star$-commutant of x^0; x^0 generates the center of the algebra.
- $\mathbb{R}^3_\lambda \simeq \bigoplus_{j\in\frac{\mathbb{N}}{2}} \mathbb{S}^j$ foliation into fuzzy spheres

Matrix basis for $\mathbb{R}^3_\lambda$

Star products of Lie algebra type have been widely studied [Gutt, Majid, Meljianac, Freidel ...]

The advantage of having a symplectic realization is that many structures may be induced (derivations, differential calculus, integration measure...)

- An important tool in applications is the existence of a matrix basis
- Moyal algebra has an orthogonal matrix basis which is given by the symbols of the operators (Wigner functions)[GraciaBondia-Varilly'89]

$|n_1, n_2\rangle\langle m_1 m_2|, \quad n_a, m_a = 0, ..., \infty$

$\left[\hat{N}_a|n_a\rangle = n_a|n_a\rangle, \quad \hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1 \quad \hat{N}_2 = \hat{a}_2^\dagger \hat{a}_2\right]$

$$\begin{aligned} f_{NM}(z, \bar{z}) &= \mathrm{Tr}\,[\hat{\Omega}(z_a, \bar{z}_a)|n_1, n_2\rangle\langle m_1 m_2|] \\ &= \frac{2}{\sqrt{N!M!\theta^{N+M}}} \bar{z}_1^{n_1} \bar{z}_2^{n_2} \star \exp(-2\bar{z}_a z_a/\theta) \star z_1^{m_1} z_2^{m_2} \end{aligned}$$

$$f_{PQ} \star_\theta f_{LM} = \delta_{QL} f_{PM}; \quad \int f_{PQ} = K\delta_{PQ}$$

Thus

$$\phi = \sum_{NM} \Phi_{NM} f_{NM}(\bar{z}, z)$$

and

$$\phi \star_\theta \psi = \sum (\Phi \cdot \Psi)_{NM} f_{NM}(\bar{z}, z)\,, \quad \int \phi = K \,\mathrm{Tr}\, \Phi$$

(infinite matrices $\longrightarrow$ regularization needed)

- In order to construct a matrix basis in $\mathbb{R}^3_\lambda$ pose $2j = n_1 + n_2, \quad m = n_1 - n_2 \qquad 2\tilde{\jmath} = p_1 + p_2, \quad \tilde{m} = p_1 - p_2$ and denote the symbols by

  $v^{j\tilde{\jmath}}_{m\tilde{m}}(z, \bar{z}), \;\; j, \tilde{\jmath} \in \mathbb{N}/2, \;\; -j \leq m \leq j,$
- for them to be in the subalgebra $\mathbb{R}^3_\lambda$ they have to star-commute with the Casimir function x_0

  $j = \tilde{\jmath} \longrightarrow v^j_{m,\tilde{m}}(x)$ orthogonal basis in $\mathbb{R}^3_\lambda$.[VW2013]

- Symbols of the matrix basis $v^j_{m\tilde{m}}, j \in \frac{\mathbb{N}}{2}, -j \le m, \tilde{m} \le j$ can be computed

$$v^j_{m\tilde{m}}(x) = \frac{e^{-2\frac{x_0}{\lambda}}}{\lambda^{2j}} \frac{(x_0+x_3)^{j+m}(x_0-x_3)^{j-\tilde{m}}\,(x_1-ix_2)^{\tilde{m}-m}}{\sqrt{(j+m)!(j-m)!(j+\tilde{m})!(j-\tilde{m})!}}$$

- Their product $v^j_{m\tilde{m}} \star v^{\tilde{j}}_{n\tilde{n}} = \delta^{j\tilde{j}}\delta_{\tilde{m}n} v^j_{m\tilde{n}}$
- $\int v^j_{m\tilde{m}} = C\delta_{m,\tilde{m}}$
- They are orthogonal
  $\int v^j_{m\tilde{m}} \star v^{\tilde{j}}_{n\tilde{n}} = \delta^{j\tilde{j}}\delta_{\tilde{m}n} \int v^j_{m\tilde{n}} = C\delta^{j\tilde{j}}\delta_{\tilde{m}n}\delta_{m,\tilde{n}}$

The star product becomes a block-diagonal infinite-matrix product

$$\phi(x) = \sum_j \sum_{m,\tilde{m}=-j}^{j} \phi^j_{m\tilde{m}} v^j_{m\tilde{m}}(x)$$

$$\begin{aligned} \phi \star \psi &= \sum \phi^j_{m_1\tilde{m}_1} \psi^j_{m_2\tilde{m}_2} v^j_{m_1\tilde{m}_1} \star v^j_{m_2\tilde{m}_2} = \sum \phi^j_{m_1\tilde{m}_1} \psi^j_{m_2\tilde{m}_2} v^j_{m_1\tilde{m}_2} \delta_{\tilde{m}_1 m_2} \\ &= \sum_{j,m_1,\tilde{m}_2} (\Phi^j \cdot \Psi^j)_{m_1\tilde{m}_2} v^j_{m_1\tilde{m}_2} \end{aligned}$$

the infinite matrix Φ is arranged into blocks
$\Phi^j = \{\phi^j_{mn}\}, -j \leq m, n \leq j$. ($\sim$ foliation of $\mathbb{R}^3$)
The integral becomes a trace

$$\int_{\mathbb{R}^3_\lambda} \phi \propto \sum_j \mathrm{Tr}_j \Phi^j$$

with Tr_j the trace in the $(2j+1) \times (2j+1)$ subspace.

Coordinate functions

We represent coordinate functions in the matrix basis

$$
\begin{aligned}
x_+ &= \sum_{j,m} \sqrt{(j+m)(j-m+1)}\, v^j_{m\,m-1} \\
x_- &= \sum_{j,m} \sqrt{(j-m)(j+m+1)}\, v^j_{m\,m+1} \\
x_3 &= \sum_{j,m} m v^j_{m\,m} \\
x_0 &= \sum_{j,m} (j+\frac{1}{2}) v^j_{m\,m}
\end{aligned}
$$

Notice that

$$x^0 \star x^0 = \sum_{j,m} (j+\frac{1}{2})^2 v^j_{m\,m}$$

different from

$$\sum_i x^i \star x^i = \sum_{j,m} j(j+1) v^j_{m\,m}$$

- Derivations are inner $D_i := ik[x^i, \cdot]_\star$
- A gauge connection is defined as previously on $\mathcal{H} = \mathbb{C} \otimes \mathbb{R}^3_\lambda$

$$\nabla_{D_i} f = \nabla_{D_i}(\mathbf{1}) \star f + D_i f \longrightarrow A_i = i\nabla_{D_i}(\mathbf{1})$$

- there is a fundamental one-form η s.t $df(D_i) = [\eta(D_i), f]$ with $\eta(D_i) = ikx^i$
- this defines a natural gauge invariant connection $\eta_i = k\delta_{ij}x^j$ with curvature $F^\eta_{ij} = 0$
- and a gauge covariant one form $\mathcal{A}_i = A_i + \eta_i$
- the curvature is $F_{ij} = (D_i A_j - D_j A_i) + [A_i, A_j] + \lambda\epsilon_{ijk} A_k = [\mathcal{A}_i, \mathcal{A}_j] + \lambda\epsilon_{ijk}\mathcal{A}_k$

The gauge action [GVW14]

As in the Moyal based 2-gauge theory discussed in last lecture, curvature is expressed in terms of the gauge covariant one form only.

It is then legitimate to assume the action to be polynomial in $\mathcal{A}$.

We consider a polynomial action in the one-form $\mathcal{A}$ which is at most quartic (being F^2 at most quartic in $\mathcal{A}$)

$$S(\mathcal{A}) = \int \left(\alpha \mathcal{A}_i \star \mathcal{A}_j \star \mathcal{A}_j \star \mathcal{A}_i + \beta \mathcal{A}_i \star \mathcal{A}_j \star \mathcal{A}_i \star \mathcal{A}_j + \gamma \varepsilon_{ijk} \mathcal{A}_i \star \mathcal{A}_j \star \mathcal{A}_k + \delta \mathcal{A}_i \star \mathcal{A}_i\right)$$

With a suitable choice of the parameters (dictated by reasonable physical requests) the action is rewritten as the sum

$$S(\mathcal{A}) = \int \left(a F_{ij} \star F_{ij} + b \varepsilon_{ijk} \mathcal{A}_i \star \mathcal{A}_j \star \mathcal{A}_k + c \mathcal{A}_i \star \mathcal{A}_i\right)$$

which is a matrix model.

We write $\mathcal{A}$ in the matrix basis

$$\sum_{j\in\frac{\mathbb{N}}{2}}\sum_{-j\leq m,\tilde{m}\leq j}(\mathcal{A}_i^j)_{m\tilde{m}}v^j_{m,\tilde{m}}$$

and replace into the action. The integral gets replaced by a trace

$$S(\mathcal{A})=\sum_j (j+1)\,\mathrm{Tr}\,_j(F_{ij}^\dagger F_{ij}+\gamma\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k+\mu\mathcal{A}_i\mathcal{A}_i)$$

This is the starting point for perturbative analysis of the gauge model.

- The action is expanded around trivial vacuum configuration.
- For a suitable choice of parameters the kinetic term is of Jacobi type.
- The propagator is found with the same techniques as in 2-d induced gauge models.
- The orthogonal polynomials which invert the kinetic term are in this case discrete dual Hahn polynomials

Results

- Within the framework described here, scalar and gauge field theories have been studied in [VW13], [GVW14] with recent results by [GPW15]
- Problems: All derivations at our disposal are "tangential" to fuzzy spheres: $D_i \propto [x_i, \cdot]_\star$
  $\Longrightarrow$ The dynamics constructed out of them is trivial in the radial direction
- The Laplacian that we proposed initially $D_i D_i$ has precisely this drawback
- Alternatives have been studied with a modified star product [JPW16]
- Good news: the models studied do not exhibit UV/IR mixing according to the perturbative analysis performed up to now.
- This is in agreement with a result by [GLV09] which states that UV/IR mixing is a generic feature of translation invariant star products

Perspectives

- The procedure can be extended to any other Lie algebra of the family described. For each of them we have a different NC algebra on $\mathbb{R}^3$.
- Much has been done for the $k - Minkowski$ case but the corresponding field theory has not been studied within this formalism.
- The $\mathbb{R}^3_\lambda$ star product is not closed though cyclic:

$$\int f \star g = \int g \star f \neq \int f \cdot g$$

- It is possible to construct a closed star product [KV16] and a non-associative closed star product [KS17]. The former has been applied in [JPW16] to get a NC gauge theory on $\mathbb{R}^3$ with a better Laplacian (triviality problem solved). The latter has promising applications in non-associative geometry and string theory.

Bibliography

[Schwinger65] J. Schwinger in Quantum theory of angular momentum" L. C. Biedenharn and H. Van Dam eds, (1965) Academic Press
[MMVZ93] V. I. Manko, G. Marmo, P. Vitale and F. Zaccaria, "A Generalization of the Jordan-Schwinger map: Classical version and its q deformation," Int. J. Mod. Phys. A **9**, 5541 (1994) [hep-th/9310053]
[GLMV02] J. M. Gracia-Bondia, F. Lizzi, G. Marmo and P. Vitale, "Infinitely many star products to play with," JHEP **0204**, 026 (2002) [hep-th/0112092]
[HLS02] A.B. Hammou, M. Lagraa and M.M. Sheikh-Jabbari, Coherent state induced star-product on $\mathbb{R}^3_\lambda$ and the fuzzy sphere, hep-th/0110291
[Kontsevich] M. Kontsevich, Lett. Math. Phys. **66** (2003) 157.
[PinzulStern] A. Pinzul and A. Stern, "Gauge Theory of the Star Product," Nucl. Phys. B **791**, 284 (2008) [arXiv:0705.1785 [hep-th]]
[VW2013] P. Vitale and J. -C. Wallet, "Noncommutative field theories on R^3_λ: Toward UV/IR mixing freedom," JHEP **1304**, 115 (2013) [arXiv:1212.5131 [hep-th]].
[GVW14] A. Géré, P. Vitale and J. C. Wallet, "Quantum gauge theories on noncommutative three-dimensional space," Phys. Rev. D **90**, no. 4, 045019 (2014) [arXiv:1312.6145 [hep-th]]
[GJW15] A. Géré, T. Juric and J. C. Wallet, "Noncommutative gauge theories on $\mathbb{R}^3_\lambda$: perturbatively finite models," JHEP **1512**, 045 (2015) [arXiv:1507.08086 [hep-th]].

[JPW16] T. Juric, T. Poulain and J. C. Wallet, "Closed star product on noncommutative $\mathbb{R}^3$ and scalar field dynamics," JHEP **1605**, 146 (2016) [arXiv:1603.09122 [hep-th]]
[KV16] V. G. Kupriyanov and P. Vitale, "Noncommutative $\mathbb{R}^d$ via closed star product," JHEP **1508**, 024 (2015) [arXiv:1502.06544 [hep-th]].
[KS17] V. G. Kupriyanov and R. J. Szabo, "G_2-structures and quantization of non-geometric M-theory backgrounds," JHEP **1702**, 099 (2017) [arXiv:1701.02574 [hep-th]]

[GLV09] S. Galluccio, F. Lizzi and P. Vitale, "Translation Invariance, Commutation Relations and Ultraviolet/Infrared Mixing," JHEP **0909**, 054 (2009) [arXiv:0907.3640 [hep-th]]