Noncommutative Gauge Theory

Patrizia Vitale

Dipartimento di Fisica Università di Napoli Federico II and INFN

Noncommutative Geometry and Applications to Quantum Physics
Quy-Nhon, Vietnam 13.7-22.7 2017

Outline

Lecture II

- Generalization of gauge connection, gauge transformation, vector bundles (from lecture I)
- UV/IR mixing for the natural QED action
- Induced Noncommutative gauge models
- Derived matrix model
- The matrix basis
- Gribov ambiguity
- Discussion

Recall that the classical action for interacting fields is

$$S[\psi, A] = S[A] + S[\psi] + S_{int}[\psi, A]$$

- It is a scalar under coordinate transformations (Poincaré)
- For it to be gauge invariant under $\psi \to \psi' = \rho(g)\psi$ (which qualifies ψ as section of a vector bundle), A, the gauge potential, has to be a connection of a principal G bundle

$$A' = gAg^{-1} + dgg^{-1}$$

with $g \in \mathcal{G} = Map(M, G)$
- and the derivative replaced by covariant derivative. Locally:

$$\partial_\mu \to \nabla_\mu = \partial_\mu + A_\mu$$

To construct a NC gauge theory we need to generalize:
- matter fields namely (the analogue of) sections of vector bundles
- (the analogue of) gauge transformations
- gauge fields namely gauge potential and curvature

Specifically for QED we have to generalize
- matter fields which are 1-d complex vector fields (sections of complex line bundle)
- $\mathcal{G} = Map(\mathbb{R}^4, U(1))$ gauge transformations
- and the $U(1)$ gauge connection

1-d complex vector bundle is replaced by 1-d complex right module over $\mathbb{R}_\theta^{2n}$ (one generator)

$$\mathcal{H} = \mathbb{C} \otimes \mathbb{R}_\theta^{2n} \sim \mathbb{R}_\theta^{2n}$$

with Hermitian structure $h : h(m_1, m_2) = m_1^\dagger \star m_2$

Gauge transformations of $\mathcal{H}$ are automorphisms of $\mathcal{H}$ compatible both with the structure of right $\mathbb{R}^{2n}_\theta$-module

$$\gamma(mf) = \gamma(m)f$$

and with the Hermitian structure

$$h(\gamma(m_1), \gamma(m_2)) = h(m_1, m_2) \quad \forall m_1, m_2 \in \mathcal{H}$$

Connection: is a linear map $\nabla : \mathrm{Der}(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$ satisfying

- $\nabla_X(m \star f) = m \star X(f) + \nabla_X(m) \star f, \nabla_{cX}(m) = c\nabla_X(m)$
- $\nabla_{X+Y}(m) = \nabla_X(m) + \nabla_Y(m) \qquad c$ in the center

Hermitian if:

$$X(h(m_1, m_2)) = h(\nabla_X(m_1), m_2) + h(m_1, \nabla_X(m_2)), \forall m_1, m_2 \in \mathcal{H}$$

- Curvature: is the linear map $F(X, Y) : \mathcal{H} \to \mathcal{H}$ defined by

$$F(X, Y)m = [\nabla_X, \nabla_Y]m - \nabla_{[X,Y]}m$$

The 1-form connection A is defined by the action on module generators:
$\nabla_X(\mathbf{f}) = \nabla_X(\mathbf{1}) \star f + \mathbf{1}X(f)$, with $\nabla_X(\mathbf{1})^\dagger = -\nabla_X(\mathbf{1})$.
$\Longrightarrow$:

- $A : X \to A(X) := \nabla_X(\mathbf{1}), \quad \forall X \in \mathrm{Der}(\mathbb{R}^{2n}_\theta)$
- $\nabla_\mu(\mathbf{1}) =: -iA(\partial_\mu) = -i\mathbf{1} \star A_\mu$
- so that
$$\nabla_\mu \mathbf{f} := \nabla_\mu(\mathbf{1} \star f) = \mathbf{1} \star (\partial_\mu f - iA_\mu \star f)$$

• Gauge transformations can be identified with the unitary elements of the module: $\mathcal{U}(\mathcal{H}) \sim \mathcal{U}(\mathbb{R}_\theta^{2n})$

Indeed

$$\gamma(\mathbf{f}) = \gamma(\mathbf{1} \star f) = \gamma(\mathbf{1}) \star f$$
$$h(\gamma(\mathbf{f}_1), \gamma(\mathbf{f}_2)) = \gamma(\mathbf{f}_1)^\dagger \star \gamma(\mathbf{f}_2) = h(\mathbf{f}_1, \mathbf{f}_2) \longrightarrow \quad \gamma(\mathbf{1})^\dagger \star \gamma(\mathbf{1}) = 1$$

we pose $\gamma(\mathbf{1}) = U \in \mathcal{U}(\mathcal{H})$

Remark Had we chosen to study non-Abelian gauge theories the 1-d complex module would have been replaced $\mathbb{C} \otimes \mathbb{R}_\theta^{2n} \to \mathbb{C}^N \otimes \mathbb{R}_\theta^{2n}$.

Properties of the gauge connection

- gauge invariance:
  $(\nabla^A_\mu)^\gamma(\phi) := \gamma(\nabla^A_\mu(\gamma^{-1}\phi)) = U \star \nabla^A_\mu U^{-1} \star \phi$
  $A^U_\mu = U \star A_\mu \star U^{-1} + iU \star \partial_\mu U^{-1}$
- Curvature:
  $F_{\mu\nu} = i([\nabla^A_\mu, \nabla^A_\nu] - \nabla^A_{[X_\mu, X_\nu]}) = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$
  $F^U_{\mu\nu} = U \star F_{\mu\nu} \star U^{-1}$

Gauge transformations $U \in \mathcal{U}(\mathcal{H})$ can be written as star exponentials
$U = \exp_\star i\alpha$ with

$$\exp_\star(f) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{f \star ... \star f}_{n \text{ times}},$$

The natural QED action

$$S = \int d^{2n}x \;\; F_{\mu\nu} \star F^{\mu\nu}$$

is gauge and Poincaré invariant but yields new pathologies w.r.t. the commutative case: UV/IR mixing, Gribov ambiguity

UV/IR mixing was first discovered for scalar field theories
[Minwalla-Van Raamsdonk-Seiberg, Chepelev-Roiban (2000)]

$$S[\phi] = \int d^{2n}x \; \partial_\mu \phi \star \partial^\mu \phi + m^2 \phi \star \phi + \lambda \phi^{\star 4}$$

then for gauge theory [Hayakawa,Matusis Susskind Toumbas(2000)]
It is about the fact that relevant physical observables (correlation functions) have a better behavior in the UV finite (as hoped) but become IR divergent. One interpretation is that, because coordinates do not commute anymore an indetermination principle is at work which mixes short and large scales.

UV/IR mixing

Alternatives have been considered which modify the classical action.
For scalar field theory, two important ones are:
Grosse-Wulkenhaar harmonic model

$$S[\phi] = \int d^{2n}x \left[\frac{1}{2}\left(\partial_\mu\phi \star \partial_\mu\phi + \Omega^2(\tilde{x}_\mu\phi) \star \tilde{x}_\mu\phi + m^2\phi \star \phi\right)\right] + \lambda\phi^{\star 4}$$

with $\tilde{x}_\mu = 2\Theta^{-1}_{\mu\nu}x^\nu$. The model is invariant under LSZ duality. It is fully renormalizable

[Grosse-Wulkenhaar '04, Rivasseau-VigneTourneret-Wulkenhaar '06]

Translation invariant model [Gurau-Magnen-Rivasseau-Tanasa '09]

$$S[\phi] = \int d^{2n}x \left[\frac{1}{2}\left(\partial_\mu\phi \star \partial_\mu\phi + \partial_\mu^{-1}\phi \star \partial_\mu^{-1}\phi + m^2\phi \star \phi\right)\right] + \lambda\phi^{\star 4}$$

with $\partial_\mu^{-1}\phi(x) := \int dp \frac{1}{ip_\mu}\tilde{\phi}(p)e^{ipx}$

For gauge theories, both prescriptions have problems. The first one breaks translation invariance, the second gauge invariance.

Let us shortly look at one alternative, which is interesting because it brings to matrix models, another important topic in NC geometry.

- Induced NC gauge models
  generalize to NC geometry an idea known in QFT and later proven with heat kernel techniques [Connes-Chamseddine, Vassilievich, Langmann*] that the log divergent term of the effective action of a field theory coupled to non-dynamical gauge fields is proportional to the dynamical action of the gauge field

* "Generalized Yang-Mills action from Dirac operator determinants" J. Math. Phys. 42 (2001) 5238

- The starting gauge invariant action is

$$\begin{aligned} S[\phi, A] &= S[\phi] + \int d^D x \Big((1+\Omega^2)\phi^\dagger \star (\tilde{x}_\mu A_\mu) \star \phi \\ &+ (1-\Omega^2)\phi^\dagger \star A_\mu \star \phi \star \tilde{x}_\mu + (1+\Omega^2 \phi^\dagger \star A_\mu \star A_\mu \star \phi \Big) \end{aligned}$$

- The effective action $\Gamma[A]$ is thus (formally) defined by

$$e^{-\Gamma[A]} = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger e^{-S[\phi, A]}$$

  which is divergent, needs to be regularized.

- The coefficients of the log divergent term yield the induced gauge action, which is

  deGoursac–Wallet–Wulkenhaar, Grosse–Wohlgenannt(2007)

  $S[A] = \int d^4 x \frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + \frac{\Omega^2}{4} \{\mathcal{A}_\mu, \mathcal{A}_\nu\}^2_\star + \kappa \mathcal{A}_\mu \star \mathcal{A}_\mu$

What is $\mathcal{A}$

- When all derivations are inner there is a fundamental one-form ξ s. t. $df(X) = [\xi(X), f]$.
  For Moyal algebra

  $$\xi = \xi_\mu dx^\mu \quad \xi_\mu := -\frac{1}{2}\tilde{x}_\mu$$

  This defines a natural gauge invariant connection.
  Indeed $\quad g \star \xi \star g^{-1} + ig \star dg^{-1} = \xi$
- The curvature of the gauge invariant connection is the constant $F^\xi_{\mu\nu} = \Theta^{-1}_{\mu\nu}$
- $\mathcal{A} = (A - \xi)$ is a gauge covariant one-form (not a connection)
- $F^A_{\mu\nu} = \Theta^{-1}_{\mu\nu} + i[\mathcal{A}_\mu, \mathcal{A}_\nu]_\star$

The Matrix model

- As a functional of $\mathcal{A}$ the action

$$S[\mathcal{A}] = \int d^4x \Big(\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + \frac{\Omega^2}{4} \{\mathcal{A}_\mu, \mathcal{A}_\nu\}_\star^2 + \kappa \mathcal{A}_\mu \star \mathcal{A}_\mu \Big)$$

  is a (infinite) matrix model (only products of $\mathcal{A}$ appear)

- In two dimensions $S[\mathcal{A}] = \int d^2x \Big((1+\Omega^2)\mathcal{A} \star \mathcal{A}^\dagger \star \mathcal{A} \star \mathcal{A}^\dagger + (3\Omega^2 - 1)\mathcal{A} \star \mathcal{A} \star \mathcal{A}^\dagger \star \mathcal{A}^\dagger + 2\kappa \mathcal{A} \star \mathcal{A}^\dagger \Big)$
  with $\mathcal{A} = \frac{\mathcal{A}_1 + i\mathcal{A}_2}{\sqrt{2}}$

- Fields can be expressed in the Moyal matrix basis [GraciaBondia–Varilly89] $\mathcal{A}(x) = \sum_{mn} \mathcal{A}_{mn} f_{mn}(x)$

The matrix basis for Moyal algebra

More convenient to go to complex coordinate functions so that $\mathbb{R}^{2n} \to \mathbb{C}^n$

and assume for simplicity $n = 1$; $z = \frac{1}{\sqrt{2}}(x_1 + i x_2)$ with star commutator

$[z, \bar{z}]_\star = \theta, \qquad z, \bar{z}$ are symbols of the operators $a, a^\dagger$

Operators are written in the so-called number basis $|n\rangle = \frac{a^{\dagger n}}{n!\theta^n}|0\rangle$

$a|n\rangle = \sqrt{n\theta}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{(n+1)\theta}|n+1\rangle$

$\hat{\phi} = \sum_{n,m} \phi_{nm}|p\rangle\langle q| \qquad n, m = 0, ..., \infty$

Then the symbol of ϕ becomes

$\phi(z, \bar{z}) = \mathrm{Tr}\, \phi \hat{U}(z, \bar{z}) \qquad \hat{U}(z, \bar{z}) = \exp(z a^\dagger - \bar{z} a) \Longrightarrow$

$$\phi(z, \bar{z}) = \sum_{n,m} \phi_{nm} f_{nm}(z, \bar{z}) \qquad \text{with}$$

$$f_{pq}(z, \bar{z}) = \mathrm{Tr}\, |p\rangle\langle q| \hat{U}(z, \bar{z}) = \frac{2}{\sqrt{p! q! \theta^{p+q}}} \bar{z}^p \star \exp(-2\bar{z}z/\theta) \star z^q$$

$f_{pq} \star f_{lm} = \delta_{ql} f_{pm} \qquad \int f_{pq} = C\delta_{pq} \qquad \int \to \mathrm{Tr}$

The Matrix model

The model has been studied in the matrix basis. A number of results is available

- Nontrivial vacuum solutions (solutions of the classical eom) have been found [de Goursac-Wallet-Wulkenhaar 2008].
- Under certain circumstances the kinetic term of the action becomes a Jacobi type operator [Martinetti-Vitale-Wallet 2013], therefore invertible for the propagator in terms of orthogonal polynomials.
- Propagator and interaction term (vertex) have been found [MVW 2013]
- Once we have the propagator and the interaction vertices, we perform a perturbative analysis of the QFT (Feynmann diagrams).

Conclusion: At least for the assumptions we have made the model has UV/IR mixing

Remember that in standard QED ($U(1)$ gauge theory) there is no Gribov ambiguity because the bundle of gauge connections is globally trivial. This means that gauge fixing succeeds in singling out one representative for each equivalence class of gauge connections.

What happens here:

Asymptotically:

$$(f \star g)(x) = f(x) \exp\left\{ \frac{i}{2} \theta^{\rho\sigma} \overleftarrow{\partial}_\rho \overrightarrow{\partial}_\sigma \right\} g(x)$$

Under the $U(1)$ gauge transformation in NCQED the gauge field A transforms as

$$A \to A'_\mu[\alpha] = U \star A_\mu \star U^\dagger + i\, U \star \partial_\mu U^\dagger, \qquad U \equiv \exp_\star(i\alpha)$$

with

$$\exp_\star(f) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{f \star \ldots \star f}_{n \text{ times}},$$

Gribov copies in NCQED

Infinitesimally

$$A \to A'_\mu[\alpha] = A_\mu + D_\mu\alpha + \mathcal{O}(\alpha)$$

where

$$D_\mu\alpha = \partial_\mu\alpha + i\,(\alpha \star A_\mu - A_\mu \star \alpha)$$

. Choose the Lorentz gauge, $\partial^\mu A_\mu = 0$ and replace for A'_μ
$\partial^\mu A'_\mu[\alpha] = 0 \to$

$$\partial^\mu D_\mu\alpha = 0$$

This is the so called Equation of copies, which may now have non trivial solutions, compared to the commutative case. Indeed it has been proven to have an infinite number of non-trivial solutions [Canfora-Kurkov-Rosa-V '16]

Summary

- We have studied NCQED on Moyal space-time
- We have found UV/IR mixing, for all models considered
- We have found Gribov ambiguity, due to noncommutativity

The failure of Moyal noncommutativity to cure (up to now) some of the pathologies of standard gauge theory justifies the search for other types of noncommutativity.

Bibliography

- S. Minwalla, M. Van Raamsdonk and N. Seiberg, "Noncommutative perturbative dynamics," JHEP **0002**, 020 (2000) doi:10.1088/1126-6708/2000/02/020 [hep-th/9912072].
- I. Chepelev and R. Roiban, "Renormalization of quantum field theories on noncommutative R^d. 1. Scalars," JHEP **0005**, 037 (2000) doi:10.1088/1126-6708/2000/05/037 [hep-th/9911098].
- H. Grosse and R. Wulkenhaar, "Renormalization of ϕ^4 theory on noncommutative $\mathbb{R}^4$ in the matrix base," Commun. Math. Phys. **256**, 305 (2005) doi:10.1007/s00220-004-1285-2 [hep-th/0401128].
- R. Gurau, J. Magnen, V. Rivasseau and A. Tanasa, "A Translation-invariant renormalizable non-commutative scalar model," Commun. Math. Phys. **287**, 275 (2009) [arXiv:0802.0791 [math-ph]].
- Chamseddine A.H. and Connes A., The spectral action principle", Commun. Math. Phys. **186**, 731 (1997)
- D. V. Vassilevich, Heat kernel expansion: Users manual, Phys. Rept. **388**, 279 (2003) hep-th/0306138
- E. Langmann Generalized Yang-Mills action from Dirac operator determinants" J. Math. Phys. 42 (2001) 5238

- H. Grosse and M. Wohlgenannt, "Induced gauge theory on a noncommutative space," Eur. Phys. J. C **52**, 435 (2007) [hep-th/0703169]

- A. de Goursac, J. C. Wallet and R. Wulkenhaar, "Noncommutative Induced Gauge Theory," Eur. Phys. J. C **51**, 977 (2007) doi:10.1140/epjc/s10052-007-0335-2 [hep-th/0703075 [HEP-TH]]
- P. Martinetti, P. Vitale and J. C. Wallet, "Noncommutative gauge theories on $\mathbb{R}^2_\theta$ as matrix models," JHEP **1309**, 051 (2013) doi:10.1007/JHEP09(2013)051 [arXiv:1303.7185 [hep-th]]

- F. Canfora, M. Kurkov, L. Rosa and P. Vitale, "The Gribov problem in

Noncommutative QED," JHEP **1601**, 014 (2016) [arXiv:1505.06342 [hep-th]].