Noncommutative Gauge Theory

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Outline

Lecture I

- What is a gauge transformation
- Commutative, Abelian and non-Abelian Gauge theories
- Local and global problems of commutative gauge theories
- Motivations for NC geometry
- Prototype of NC algebra: Moyal algebra
- Derivation based differential calculus

Lecture II

Noncommutative Moyal gauge theories

Lecture III

Noncommutative Lie algebra type gauge theories

General setting for first part of this lecture:

- M smooth manifold (space-time, rigorously, should be compact)
- G finite dimensional Lie group
- $\pi : P \to M$ principal G-bundle

Gauge theories are theories where dynamical variables are connections of G-bundles.

- They describe fundamental interactions
- G qualifies the interaction ($U(1)$ for Electromagnetic interaction, $SU(2) \times U(1)$ electroweak interactions, $SU(3)$ strong interactions)
- The number of generators of G is equal to the number of particles which mediate the interaction (so called vector bosons)
- For Electrodynamics this is 1 (the foton)
- Gauge transformations are special automorphsims of P which are symmetries of the dynamics

Definition Aut(P)

An *automorphism* of P is a *diffeomorphism* $\varphi : P \to P$ which is G-equivariant, that is $\varphi(p \cdot g) = \varphi(p) \cdot g$ for all $p \in P$ and $g \in G$.

Every $\varphi \in \mathrm{Aut}(P)$ induces a diffeomorphism $\tilde{\varphi}$ on the basis manifold.
Locally $\quad \phi : p \simeq (x, g(x)) \longrightarrow \quad \phi(p) \simeq (\tilde{\phi}(x), g'(x))$

The map H which associates $\tilde{\varphi} \in$ Diff (M) to $\varphi \in \mathrm{Aut}(P)$ is a *group homomorphism*:

$$H : \phi \in Aut(P) \to \tilde{\phi} \in Diff(M)$$

$$H(\phi \cdot \psi) = H(\phi) \circ H(\psi)$$

Definition $\mathcal{G}(P)$

The gauge group of P is $\mathcal{G}(P) :=\ker(H)$. Its elements are called *gauge transformations* or also vertical automorphisms.

$$\phi \in \mathcal{G}(P) \subset Aut(P) \text{ if } \pi(\phi(p)) = \pi(p)$$

that is $\quad \tilde{\phi} = id \in Diff(M)$

$\longrightarrow$

Physics:

$\mathcal{G} \simeq \mathrm{Map(M \to G) = \{smooth\ maps\ g : x \in M \in G\}}$

- Physical request when $M = \mathbb{R}^n$: $g(x) \to 1 \ as \ |x| \to \infty$
- Physical states of matter fields are invariant only if this is assumed
- $\mathbb{R}^n \to S^n$ (origin of the Gribov problem in non-Abelian gauge theories)

- $\mathcal{A}=$ space of gauge connections of (P, M, G), locally maps $A: M \to \Omega^1(M) \otimes \mathfrak{g}$

$$A^g = gAg^{-1} + dgg^{-1}$$

- Matter fields coupled to interactions are sections of associated vector (spinor) bundles $(E(P), V, M, G)$
- A gauge connection on P induces a connection and a covariant derivative for sections $\psi : M \to E$

$$\nabla : Der(\Gamma(E)) \times \Gamma(E) \to \Gamma(E)$$

which obeys

$$\nabla_X(f\psi) = X(f)\psi + f\nabla_X(\psi)$$
$$\nabla_{fX}(\psi) = f\nabla_X(\psi) \qquad f \in \mathcal{F}(M)$$
$$\nabla_{X+Y}(\psi) = \nabla_X(\psi) + \nabla_Y(\psi)$$
$$\nabla_X(\psi + \phi) = \nabla_X(\psi) + \nabla_X(\phi)$$

- Locally $\quad \nabla_\mu f_\psi = \partial_\mu f_\psi + A \cdot f_\psi$

Classical Euclidean action for gauge fields

- A pure theory of fundamental interactions (no matter fields) is a theory where the dynamical fields are the gauge connections
- $\mathcal{B} = \mathcal{A}/\mathcal{G}$ space of physical configurations

$$S[A] = Tr \int_M F \wedge \star_H F = \frac{1}{4} Tr \int F_{\mu\nu} F^{\mu\nu} d\Omega$$

with $\quad F = dA + A \wedge A \in \Omega^2(M) \otimes \mathfrak{g}$

- Lie algebra $\quad \mathfrak{g} = \mathfrak{u}(N), \quad \mathfrak{su}(N) \quad$ or products (Standard Model), Lorentz group (general relativity) ...

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$$S[A] = \frac{1}{2} \int A^a_\mu M^{\mu\nu}_{ab} A^b_\nu d\Omega$$

$$M^{\mu\nu}_{ab} = \delta_{ab}(-\Box \delta^{\mu\nu} + \partial^\mu \partial^\nu)$$

not invertible: $\partial_\nu g g^{-1}$ is eigenvector of $M^{\mu\nu}$ with zero eigenvalue

- For Electrodynamics $\mathfrak{g} = \mathfrak{u}(1) \qquad g(x) = \exp \alpha(x)$

$A^g = gAg^{-1} + dgg^{-1} \quad \rightarrow \quad A^g_\mu = A_\mu + \partial_\mu \alpha \quad M^{\mu\nu} \partial_\mu \alpha = 0$

- Quantum theory:

  Classical action $S[A]$ $\rightarrow$ Quantum action $\Gamma[\mathbb{A}]$
- $\Gamma[\mathbb{A}] = Leg(\ln Z[J])[\mathbb{A}]$

$$Z[J] = \int [d\mu(\mathcal{A})] exp\left(-S[A] + J \cdot A\right)$$

- But we can't perform the Gaussian integral unless we fix the gauge

$$\delta(f(A) - h)$$

and restrict the integration to equivalence classes

$$[d\mu(\mathcal{A})] \rightarrow [d\mu(\mathcal{A}/\mathcal{G})]$$

This amounts to choose a surface $\Sigma_f \subset \mathcal{A}$ which possibly intersects the gauge orbits only once: a section for the principal bundle

$$\begin{array}{ccc} \mathcal{A}(P) & \leftarrow & \mathcal{G} \\ \downarrow & & \\ \mathcal{B}(P) & & \end{array}$$

The choice of Σ is the gauge fixing; for example $d \star_H A = 0$ $(\partial_\mu A^\mu = 0)$.
The classical action is invariant $\longrightarrow$ insensitive to the gauge fixing.

How to restrict the integration to equivalence classes, namely how to obtain $[d\mu(\mathcal{B})]$

Locally (ignore global issues for the moment) $\mathcal{A} \sim \mathcal{B} \times \mathcal{G} \quad \Longrightarrow$

$$[d\mu(\mathcal{A})] = [d\mu(\mathcal{B})]\ [d\mu(\mathcal{G})]$$

for gauge transformations close to the identity $[d\mu(\mathcal{G})] \simeq [d\alpha]$

To perform the change of variable $[d\alpha] \to [df(A)]$:
insert the Jacobian

$$\mathrm{Det}^{ab}\Delta_{\mathrm{FP}}(x,y) = \mathrm{Det}\frac{\delta f^a(x)}{\delta\alpha^b(y)} \quad \Longrightarrow$$

$$[d\mu(\mathcal{A})]\mathrm{Det}\Delta = [[d\mu(\mathcal{B})][d\alpha]\mathrm{Det}\Delta = [d\mu(\mathcal{B})]\ [df]$$

and integrate over [df] with the delta function:

$$[d\mu(\mathcal{A})]\ \mathrm{Det}\Delta\ \ \delta(f(A) - h(x)) = [d\mu(\mathcal{B})]$$

.

Back to global approach

- If we assume $\mathcal{A}$ globally trivial bundle $\rightarrow \Pi_j(\mathcal{A}) = \Pi_j(\mathcal{G}) + \Pi_j(\mathcal{B})$
- $\mathcal{A}$ is an affine space

$$A_\tau = (1-\tau)A_1 + \tau A_2 \quad 0 \leq \tau \leq 1$$

with

$$A_\tau^g = gA_\tau g^{-1} + dgg^{-1}$$

$\Longrightarrow \quad \mathcal{A}$ homotopically trivial

- $\mathcal{G} = \mathrm{Map}(\mathrm{S}^4, \mathrm{G}) \qquad (g(x) \rightarrow 1, \ \ |x| \rightarrow \infty)$

$\Longrightarrow \Pi_1(\mathcal{G}) = \{g : S^5 \rightarrow G\}$

$$\Pi_1(\mathcal{G}) = \Pi_5(G)$$

for

$G = U(N)$ $\Pi_5 = Z$, $N \geq 3$; $\Pi_5 = Z_2$, $N = 2$; $\Pi_5 = 0, \ N = 1$

Topological obstructions

$\Longrightarrow \mathcal{G}$ only homotopically trivial for QED $G = U(1)$.

$\Longrightarrow \qquad \mathcal{A} \neq \mathcal{G} \times \mathcal{B}$ for $G = U(N), N \geq 2$.

Summary: Non-Abelian gauge theories do not admit global sections
Physics: gauge fixing doesn't single out one representative for each equivalence class for non-Abelian gauge theories (Gribov ambiguity Gribov, Singer, Narasimhan and Ramadas '78-'79)

Problems of QFT:
- UV divergences $\rightarrow$ short distance cut-off needed, renormalization;
- IR divergences $\rightarrow$ Gribov-Zwanziger-Dell'Antonio modification of propagator

What is the situation for NCQFT?

Motivations for NCG:

- Gravity motivations:
  - Gedanken experiments putting together general relativity and quantum mechanics are not compatible with pseudo-Riemannian space-time [DFR94];
  - Loop Quantum Gravity has discrete spectrum for geometric operators
- Regularization of Quantum Field Theory
- Low energy regimes of strings with background B field. Double field theory (Manifest duality invariance).

- The simplest noncommutative space is modeled on the phase space of quantum mechanics:
  - First, go to dual description in terms of algebra of functions on classical phase space
  - Quantize (make it "noncommutative phase space")
  - No smooth manifold anymore
  - Noncommutativity can be described in terms of a star product: quantum mechanics in the Moyal approach $\rightarrow$ *blackboard*
- Do the same for space-time $\rightarrow [\hat{x}_i, \hat{x}_j] = i\theta_{ij}$

  - θ constant

  - replace operators with an algebra of functions on space-time (assume it even-dim.), with noncommutative star product

For coordinate functions

$$x_i \star x_j - x_j \star x_i = i\theta_{ij}$$

The Moyal star-product (asymptotic form):

$$(f \star g)(x) = f(x) \exp\left\{ \frac{i}{2} \theta^{\rho\sigma} \overleftarrow{\partial}_\rho \overrightarrow{\partial}_\sigma \right\} g(x)$$

Notice:

$$x^i \star f - f \star x^i = i\,\theta^{ij} \partial_j f$$

$$\int_{\mathbb{R}^n} f \star g = \int_{\mathbb{R}^n} g \star f \quad \left(= \int_{\mathbb{R}^n} f \cdot g \right)$$

the integral is a trace

$(\mathcal{F}(R^{2n}), \star_\theta) =: \mathbb{R}^{2n}_\theta$ is the Moyal algebra

- It is the associative algebra $\{\mathcal{L} \cap \mathcal{R} = T \in \mathcal{S}' : T \star f \in \mathcal{S}, f \star T \in \mathcal{S}, \forall f \in \mathcal{S}\}$
- The star product is defined for Schwartz functions on $\mathbb{R}^{2n}$

$$f \star g(x) = \frac{1}{(\pi\theta)^{2n}} \int f(x+y) g(x+z) e^{-2iy^\mu \Theta^{-1}_{\mu\nu} z^\nu}$$

- extended to $\mathcal{S}' \times \mathcal{S}$ $\langle T \star f, h \rangle = \langle T, f \star h \rangle$ (sim. $\mathcal{S} \times \mathcal{S}'$)
- and to $\mathcal{S}' \times \mathcal{S}'$ $\langle T \star T', h \rangle = \langle T, T' \star h \rangle$ Θ is block diagonal, antisymmetric with θ real.

$$\Theta = \theta \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & .. \end{pmatrix}$$

The Moyal algebra $\mathbb{R}^{2n}_\theta$

- $\mathcal{S}(\mathbb{R}^{2n})$ Schwartz functions
- $\mathcal{S}'(\mathbb{R}^{2n})$ dual algebra of tempered distributions
- $\mathcal{L}, \mathcal{R} \subset \mathcal{S}'$ left and right multiplier algebras
- $\mathbb{R}^{2n}_\theta$ is unital and involutive. It contains $\mathcal{S}$, polynomials, constants [Varilly, Gracia-Bondia, Soloviev arxiv-1012.0669]

$$f \star_\theta g(x) = \exp\left(\frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial w^\mu}\right) f(y)g(w)|_{y=w=x}$$

$$[x^\mu, x^\nu]_{\star_\theta} = i\theta^{\mu\nu}$$

which describes space-time noncommutativity and implies the presence of a minimal area $\simeq \theta$

Minimal derivation based differential calculus over the Moyal algebra. [DV88, DVM94, M08, W09, CMW11, MVZ]

For an associative algebra a differential calculus can always be defined algebraically [IS67, LM90], once a Lie algebra of derivations, $\mathcal{L}$, is given:
- 1-forms α are linear maps from $\mathcal{L}$ to $\mathcal{A}$.
- The exterior derivative d is defined as

$$d\alpha(X, Y) = \rho(X)(\alpha(Y)) - \rho(Y)(\alpha(X)) - \alpha([X, Y])$$

- If $\rho : \mathcal{L} \to Der(\mathcal{A})$ is a Lie algebra homomorphism, then $d^2 = d \circ d$ is zero.
- Higher forms are defined as skew-symmetric multilinear maps from $\mathcal{L}$ to the associative algebra $\mathcal{A}$.

We do the same for noncommutative algebras

$\partial_\mu = -i\theta^{-1}_{\mu\nu}[x^\nu, \cdot]_\star$ generate the minimal Lie algebra of derivations of $\mathbb{R}^{2n}_\theta$

Other derivations can be chosen (for example the whole inhomogeneous simplectic group $ISp(4, \mathbb{R})$ or subalgebras)

- inner
- not a left module because

$$f \star \partial_\mu (g \star h) \neq f \star \partial_\mu g \star h + g \star f \star \partial_\mu h$$

- d, i_{∂_μ} defined algebraically,

$$df \wedge_\star dg(X, Y) = f(X) \star g(Y) - f(Y) \star g(X);$$

- forms are constructed by duality:

$\Omega^0 \equiv \mathbb{R}^{2n}_\theta$

$\Omega^1 \;\; g \star df(X) = g \star X(f), \quad i_X\alpha = \alpha(X)$

$\Omega^2 : f \star d\alpha(X_1, X_2) := f \star \Big(X_1(\alpha(X_2)) - X_2(\alpha(X_1)) - \alpha([X_1, X_2])\Big)$

...analogously Ω^n.

They are left $\mathbb{R}^{2n}_\theta$ modules

- derivations have to be "sufficient":

$$df(X_\mu) = 0 \;\forall\mu \to f \text{ is central}$$

Summary

- We have reviewed standard gauge transformations in a language which can be generalized to NC setting
- We have introduced NC space time of Moyal type
- Derivation based differential calculus

In Lecture II we shall define NC gauge connection and NC gauge transformations and we shall apply to NC gauge theory of Moyal type

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