Introduction to non-commutative analysis and integration

Fedor Sukochev

July 20, 2017

Pseudo-differential operators (first 3 slides is a recollection)

Definition

An operator of the form

$$(Px)(t) = \int_{\mathbb{R}^d} p(t,u) e^{i\langle t,u\rangle} (\mathcal{F}x)(u) du.$$

for sufficiently good function p is called pseudo-differential.

If

$$p = \sum_{n\geq 0} p_n,$$

where $p_n(t,u)$ is the homogeneous (except for some neighbourhood of 0) function of u of order $-n$, then the function p_0 is called a principal symbol of a pseudo-differential operator P.
We say that P is of order $-n$ if $p_k = 0$, $0 \leq k < n$.

Why the principal symbol is important?

The most important feature of the principal symbol mapping is that

$$p_0(PQ) = p_0(P)p_0(Q)$$

for sufficiently good operators P and Q.
So, principal symbol is a homomorphism.

Question

Which algebra of operators is the natural domain for the principal symbol?

Answer

We construct a principal symbol as a continuous $*-$homomorphism of C^*-algebras.

Original Connes Trace Formula

Let M be a $d-$dimensional compact Riemannian manifold and let P be a (uniform classical) pseudo-differential operator of order 0 on M. Let Δ_M be the Laplace-Beltrami operator. The assertion below is due to Connes [Connes-action].

Theorem

For every P as above and for every Dixmier trace tr_ω, we have

$$\mathrm{tr}_\omega(P(1-\Delta_M)^{-\frac{d}{2}}) = \mathrm{Res}_W(P).$$

Here, $\mathrm{Res}_W(P)$ is the integral of principal symbol of P (defined yesterday over co-sphere bundle).

Setting

Let $D_k = \frac{\partial}{i\partial t_k}$ be the $k-$th partial derivative operator on $\mathbb{R}^d$ (these are unbounded self-adjoint operators on $L_2(\mathbb{R}^d)$). In what follows, $\nabla = (D_1, \cdots, D_d)$ and $\Delta = \sum_{k=1}^d \frac{\partial^2}{\partial^2 t_k} = -\sum_{k=1}^d D_k^2$. Let the $d-$dimensional vector $\frac{\nabla}{(-\Delta)^{\frac{1}{2}}}$ be defined by the functional calculus.

Domain of a principal symbol (Dao-1)

We construct a generalised principal symbol mapping as a bounded $*$-homomorphism of C^*-algebras.
Let $\pi_1 : L_\infty(\mathbb{R}^d) \to B(L_2(\mathbb{R}^d))$ and $\pi_2 : L_\infty(\mathbb{S}^{d-1}) \to B(L_2(\mathbb{R}^d))$ be the unital $*-$representations given by the formulae

$$\pi_1(f) = M_f, \quad \pi_2(g) = g(\frac{\nabla}{(-\Delta)^{\frac{1}{2}}}).$$

Let Π be the C^*-subalgebra in $B(L_2(\mathbb{R}^d))$ generated by the algebras $\pi_1(L_\infty(\mathbb{R}^d))$ and $\pi_2(L_\infty(\mathbb{S}^{d-1}))$.
In what follows, we denote by $L_\infty(\mathbb{R}^d)\bar{\otimes}L_\infty(\mathbb{S}^{d-1})$ be the weak closure of the algebraic tensor product $L_\infty(\mathbb{R}^d) \otimes L_\infty(\mathbb{S}^{d-1})$ with respect to the representation $\pi_1 \otimes \pi_2$. We identify $L_\infty(\mathbb{R}^d)\bar{\otimes}L_\infty(\mathbb{S}^{d-1})$ with the algebra $L_\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$.

New notion of a principal symbol

Theorem

There exists a unique norm-continuous $*-$homomorphism $\mathrm{symb}:\Pi\to L_\infty(\mathbb{R}^d)\bar{\otimes}L_\infty(\mathbb{S}^{d-1})$ such that

$$\mathrm{symb}(\pi_1(f))=f\otimes 1,\quad f\in L_\infty(\mathbb{R}^d),$$

$$\mathrm{symb}(\pi_2(g))=1\otimes g,\quad g\in L_\infty(\mathbb{S}^{d-1}).$$

We call symb a principal symbol mapping.

Connes Trace Formula in terms of symb mapping

We say that $T \in \Pi$ is compactly based if there is a function ϕ with compact support such that $T\pi_1(\phi) = T$. Our version of Connes Trace Theorem (for $\mathbb{R}^d$) reads as follows.

Theorem

If $T \in \Pi$ is compactly based, then

$$\varphi(T(1-\Delta)^{-\frac{d}{2}}) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{symb}(T)$$

for every continuous normalised trace φ on $\mathcal{L}_{1,\infty}$.

How our result extends the original one?

Let P be a (uniform classical) compactly based pseudo-differential operator of order 0 on $\mathbb{R}^d$ and let p be its principal symbol. Let

$$(Tx)(t) = \int_{\mathbb{R}^d} p(t, \frac{u}{|u|}) e^{i\langle t,u\rangle} (\mathcal{F}x)(u) du.$$

We have that $T \in \Pi$ and $\mathrm{symb}(T) = p$.
Moreover, $T - P \in \mathcal{L}_{d,\infty}$ and

$$T(1-\Delta)^{-\frac{d}{2}} - P(1-\Delta)^{-\frac{d}{2}} \in \mathcal{L}_{d,\infty} \times \mathcal{L}_{1,\infty} \subset \mathcal{L}_1.$$

Since φ vanishes on $\mathcal{L}_1$, it follows from *our version* of Connes Trace Formula that

$$\varphi(P(1-\Delta)^{-\frac{d}{2}}) = \varphi(T(1-\Delta)^{-\frac{d}{2}}) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{symb}(T) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} p.$$

Principal symbol in the noncommutative geometry–DAO-2

We consider two unital C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with faithful representations π_1 and π_2 respectively on the same Hilbert space H. We define $\Pi(\mathcal{A}_1, \mathcal{A}_2)$ to be the C^*-subalgebra generated by $\pi_1(\mathcal{A}_1)$ and $\pi_2(\mathcal{A}_2)$ of the $*$-algebra of all bounded operators on H. We discuss conditions on $\mathcal{A}_1$, $\mathcal{A}_2$, π_1 and π_2 which allow the existence of a "principal symbol map",

$$\mathrm{sym} : \Pi(\mathcal{A}_1, \mathcal{A}_2) \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$$

where $\otimes_{\min}$ denotes the minimal C^*-norm on the algebraic tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$. We verify these conditions in a number of important cases, in particular for noncommutative tori and noncommutative Euclidean spaces.

Principal symbol in the noncommutative geometry–DAO-2

We introduce an abstract theory of pseudodifferential operators extending the results of [DAO-1]. The advantage of this generality is that we can define algebras of pseudodifferential operators in entirely new settings: in particular for noncommutative tori and noncommutative Euclidean space, commutative and noncommutative spheres, quantum groups (like $\mathrm{SU}_q(2)$). Let $\mathcal{A}_1$, $\mathcal{A}_2$ be the algebras in any of our examples. We establish that for any continuous normalised trace φ on $\mathcal{L}_{1,\infty}$, and $T \in \Pi(\mathcal{A}_1, \mathcal{A}_2)$,

$$\varphi(T(1-\Delta)^{-d/2}) = c(\mathcal{A}_1, \mathcal{A}_2)\left(\tau_\theta \otimes \int_{\mathbb{S}^{d-1}}\right)(\mathrm{sym}(T)).$$

and $c(\mathcal{A}_1, \mathcal{A}_2)$ is a nonzero constant depending on the choices of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Hochschild cycles

The Hochschild boundary $b:\mathcal{A}^{\otimes(n+1)}\to\mathcal{A}^{\otimes n}$ is defined by setting

$$b(a_0\otimes\cdots\otimes a_n)=a_0a_1\otimes a_2\otimes\cdots\otimes a_n+$$

$$+\sum_{k=1}^{n-1}(-1)^k a_0\otimes a_1\otimes\cdots a_{k-1}\otimes a_ka_{k+1}\otimes a_{k+1}\otimes\cdots\otimes a_n+$$

$$+(-1)^n a_na_0\otimes a_1\otimes a_2\otimes\cdots\otimes a_{n-1}.$$

If $c\in\mathcal{A}^{\otimes(n+1)}$ is such that $bc=0$, then c is called a Hochschild cycle. For example, if a_0 commutes with a_1, then $a_0\otimes a_1$ is a Hochschild cycle.

Hochschild class of the Chern character

Let $(\mathcal{A}, H, D)$ be $p-$dimensional spectral triple. Define multilinear mappings $ch : \mathcal{A}^{\otimes(p+1)} \to \mathcal{L}(H)$ and $\Omega : \mathcal{A}^{\otimes(p+1)} \to \mathcal{A}$ by setting

$$ch(a_0 \otimes \cdots \otimes a_p) = F\Gamma \prod_{k=0}^{p} [F, a_k], \quad \Omega(a_0 \otimes \cdots \otimes a_p) = \Gamma a_0 \prod_{k=1}^{p} [D, a_k].$$

Connes-Chern character $\mathcal{A}^{\otimes(p+1)} \to \mathbb{C}$ is defined by setting

$$Ch(c) = \frac{1}{2}\mathrm{Tr}(ch(c)), \quad c \in \mathcal{A}^{\otimes(p+1)}.$$

Connes Character Theorem

The following is the most general form of Connes Character Theorem as proved in [CRSZ].

Theorem

Let $(\mathcal{A}, H, D)$ be $p-$dimensional spectral triple and let $c \in \mathcal{A}^{\otimes (p+1)}$ be a Hochschild cycle. For every normalised trace on $\mathcal{L}_{1,\infty}$, we have

$$\varphi(\Omega(c)(1 + D^2)^{-\frac{p}{2}}) = Ch(c).$$

Application in Connes Reconstruction Theorem. I

Spectral triple in Connes Reconstruction Theorem is commutative. Hence, if c is a Hochschild cycle, then so is $(x \otimes 1^{\otimes p}) \cdot c$ for every $x \in \mathcal{A}$.
One of the conditions in Connes Reconstruction Theorem is the existence of a Hochschild cycle c such that $\Omega(c) = 1$ (verification of this condition for manifolds is standard). Now, we have

$$\Omega((x \otimes 1^{\otimes p}) \cdot c) = x \cdot \Omega(c) = x.$$

Application in Connes Reconstruction Theorem. II

Applying Connes Character Theorem to the cycle $(x \otimes 1^{\otimes p}) \cdot c$, we obtain

$$\varphi(x(1+D^2)^{-\frac{p}{2}}) = Ch((x \otimes 1^{\otimes p}) \cdot c), \quad x \in \mathcal{A}.$$

In other words, $x(1+D^2)^{-\frac{p}{2}}$ is universally measurable and we have a universal integral given by the formula

$$x \to Ch((x \otimes 1^{\otimes p}) \cdot c), \quad x \in \mathcal{A}.$$

This observation is the first step in the proof of Connes Reconstruction Theorem.

References

A. Connes, *The action functional in noncommutative geometry,* Comm. Math. Phys. **117(4)**, (1988) 673–683.

A. Connes. *On the spectral characterization of manifolds.* arxiv 0810.2088, 10 2008. URL http://arxiv.org/abs/0810.2088v1.

A. Carey, A. Rennie, F. S., and D. Zanin *Universal measurability and the Hochschild class of the Chern character*, J. Spectr. Theory 6 (2016), 1–41.

F. S. and D. Zanin. *Dao of the principal symbol I*. Submitted manuscript.

E. Mcdonald, F. S. and D. Zanin. *Dao of the principal symbol II*. Submitted manuscript.