Introduction to non-commutative analysis and integration

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Symbols

Definition

Let $m \in \mathbb{R}$. A function $p \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ satisfying the condition

$$\sup_{x,s} |\partial_x^\alpha \partial_s^\beta p(x,s)|(1+|s|^2)^{\frac{1\cdot|\beta|+0\cdot|\alpha|-m}{2}} < \infty$$

for every multi-indices $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$ is called a symbol of order m.

Pseudo-differential operators (PDOs)

By $\mathcal{S}(\mathbb{R}^d)$ we denote the space of Schwartz functions (the smooth functions of rapid decay).

Definition

Let $m \in \mathbb{R}$ and let p be a symbol of order m. The operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ given by the formula

$$(Au)(x) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\langle x-y,s\rangle} p(x,s)u(y)\ dy\ ds,\ \ u \in \mathcal{S}(\mathbb{R}^d)$$

is called a pseudo-differential operator of order m.

Classical PDOs

Definition

A pseudo-differential operator A of order m is called classical if its symbol has an asymptotic expansion

$$p \sim \sum_{j=0}^{\infty} p_{m-j},$$

where each $p_{m-j} := p_{m-j}(x, s)$ is a symbol of order $m - j$ and is a homogeneous function of order $m - j$ in the variable $s \in \mathbb{R}^d$ except near zero. The function p_m is called the principal symbol of a classical pseudo-differential operator of order m. In particular, if A and B are classical pseudo-differential operators of orders m_1 and m_2, then $C = AB$ is a (well-defined) classical pseudo-differential operator of order $m_1 + m_2$ with principal symbol given by the product of the principal symbols of A and B .

Compactly supported PDOs

For a smooth function with compact support $\phi \in C_c^\infty(\mathbb{R}^d)$ we define the multiplication operator $(M_\phi f)(x) = \phi(x)f(x)$, $f \in \mathcal{S}(\mathbb{R}^d)$.

Definition

A pseudo-differential operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is said to be compactly supported if $M_\phi A M_\psi = A$ for some $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$.

Pseudo-differential operators $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ associated to the class of symbols of order $m < 0$ generally do not extend to a compact linear operators $A : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$.
Mariusz Wodzicki established the existence of a distinct trace on the classical compactly supported pseudo-differential operators of order d, called the noncommutative residue. The noncommutative residue of Wodzicki is defined by integrating the principal symbol of the classical pseudo-differential operator over the co-sphere bundle on $\mathbb{R}^d$

Connes' Trace Theorem

Theorem

Every compactly supported classical pseudo-differential operator $A : C_c^\infty(\mathbb{R}^d) \to C_c^\infty(\mathbb{R}^d)$ of order $-d$ extends to a compact linear operator belonging to $\mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$ and

$$\mathrm{Tr}_\omega(A) = \frac{1}{d(2\pi)^d}\mathrm{Res}_W(A),$$

where

$$\mathrm{Res}_W(A) := \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} p_{-d}(x,s)ds\,dx$$

is Wodzicki's (noncommutative) residue of A and Tr_ω is any Dixmier trace.

Connes' statement was given for closed manifolds, but it is equivalent to this Theorem. One of the reasons for generalising Connes' trace theorem is to understand traces of PDOs of order $-d$ that are not classical PDOs.

Connes' Trace Theorem–discussion

Connes' trace theorem, as it is known, has become the cornerstone of noncommutative integration in noncommutative geometry. Applications of Dixmier traces as the substitute noncommutative residue and integral in non-classical spaces range from fractals, to foliations, to spaces of noncommuting co-ordinates (Moyal spaces), and applications in string theory and Yang-Mills, , Einstein-Hilbert actions and particle physics' standard model. Connes' trace theorem, though, is not complete. There are other traces, besides Dixmier traces, on the ideal of compact operators whose singular values are $O(n^{-1})$. Wodzicki showed that the noncommutative residue is essentially the unique trace on classical pseudo-differential operators of order $-d$, so it should be expected that every suitably normalised trace computes the noncommutative residue. Also, all pseudo-differential operators have a notion of principal symbol and Connes' trace theorem opens the question of whether the principal symbol of non-classical operators can be used to compute their Dixmier trace.

Laplacian modulated operators

Let $\mathcal{L}_2$ denote the class of Hilbert-Schmidt operators on the Hilbert space $L_2(\mathbb{R}^d)$. Let

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$

be the Laplacian on $\mathbb{R}^d$.

Definition

Let $d \in \mathbb{N}$. A bounded operator $A : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is called Laplacian modulated if

$$\sup_{t>0} t^{1/2} \|A(1 + t(1-\Delta)^{-d/2})^{-1}\|_{\mathcal{L}_2} < \infty.$$

Symbols of Laplacian modulated operators

If follows from the definition that every Laplacian modulated operator A is Hilbert-Schmidt, so it has a unique symbol in $L_2(\mathbb{R}^d, \mathbb{R}^d)$ denoted by p_A.
By Theorem 11.3.17[1] for every compactly supported pseudo-differential operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ of order $-d$ its extension to a compact linear operator $A : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is Laplacian modulated.
According to Remark 11.3.14[1] an operator A on $L_2(\mathbb{R}^d)$ is Laplacian modulated if and only its symbol p_A satisfies the condition

$$\sup_{t>0}(1+t)^{d/2}\left(\int_{|s|>t}\int_{\mathbb{R}^d}|p_A(x,s)|^2\,dxds\right)^{1/2}<\infty. \tag{1}$$

[1] S. Lord, F. Sukochev, and D. Zanin. *Singular Traces: Theory and Applications*, volume 46 of *Studies in Mathematics*. De Gruyter, 2012.

Vector-valued residue of Laplacian modulated operators

It was shown in Proposition 11.3.18[1] that, for every compactly supported Laplacian modulated operator A with symbol p_A, the sequence

$$\left\{ \frac{1}{\log(2+n)} \int_{\mathbb{R}^d} \int_{|s| \le n^{1/d}} p_A(x,s) ds\, dx \right\}_{n \ge 0}$$

is bounded. Therefore, the following definition makes sense.

Definition

The linear map

$$A \mapsto \mathrm{Res}(A) := \left[\frac{1}{\log(2+n)} \int_{\mathbb{R}^d} \int_{|s| \le n^{1/d}} p_A(x,s) ds\, dx \right]$$

from the set of all compactly supported Laplacian modulated operators to ℓ_∞ / c_0 is called the residue, where $[\cdot]$ denotes the equivalence class in ℓ_∞ / c_0.

Vector-valued residue and the Wodzicki Residue

Proposition

Let P be a compactly supported classical pseudo-differential operator of order $-d$. We have that $\mathrm{Res}(P)$ is convergent and

$$\lim_{n\to\infty} (\mathrm{Res}(P))_n = \mathrm{Res}_W(P).$$

We identify the equivalence classes of constant sequences in ℓ_∞/c_0 with scalars. In the case that $\mathrm{Res}(P)$ is the class of a constant sequence, then we say that $\mathrm{Res}(P)$ is a scalar and identify it with the limit of the constant sequence.
Thus the vector-valued residue Res is the extension of Wodzicki's residue Res_W.

Connes' Trace Theorem-1

Let $P : C_c^\infty(\mathbb{R}^d) \to C_c^\infty(\mathbb{R}^d)$ be a compactly based pseudo-differential operator of order $-d$ with residue $\mathrm{Res}(P)$. Then (the extension) $P : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ belongs to $\mathcal{L}_{1,\infty}$ and:

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d}\omega(\mathrm{Res}(P)), \quad \forall \mathrm{Tr}_\omega;$$

Connes' Trace Theorem-1

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d}\mathrm{Res}(P), \quad \forall \mathrm{Tr}_\omega;$$

iff

$$\int_{\mathbb{R}^d}\int_{|\xi|\leq n^{1/d}} p(x,\xi)d\xi\, dx = \frac{1}{d}\mathrm{Res}(P)\,\log n + o(\log n)$$

for a scalar $\mathrm{Res}(P)$;

$$\tau(P) = \frac{\tau\circ\mathrm{diag}\left(\{\frac{1}{n}\}_{n=1}^\infty\right)}{d(2\pi)^d}\mathrm{Res}(P)$$

for every trace τ on $\mathcal{L}_{1,\infty}$ iff

$$\int_{\mathbb{R}^d}\int_{|\xi|\leq n^{1/d}} p(x,\xi)d\xi\, dx = \frac{1}{d}\mathrm{Res}(P)\,\log n + O(1)$$

for a scalar $\mathrm{Res}(P)$.

Connes' Trace Theorem-1-Application

Theorem (Integration of square integrable functions)

If $f \in L_2(\mathbb{R}^d)$ has compact support then $M_f(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}$ is Laplacian modulated and such that

$$\tau(M_f(1-\Delta)^{-d/2}) = \frac{\mathrm{Vol}\mathbb{S}^{d-1}}{d(2\pi)^d} \int_{\mathbb{R}^d} f(x)dx$$

for every trace τ on $\mathcal{L}_{1,\infty}$ with $\tau(\mathrm{diag}\{k^{-1}\}_{k=1}^{\infty}) = 1$.

The same statement can be made for closed manifolds, omitting of course the requirement for compact support of f, and with the Laplace-Beltrami operator in place of the ordinary Laplacian

Connes' Trace Theorem-2

Theorem

Let A be a compactly supported Laplacian modulated operator with symbol p_A. We have $A \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$. Moreover,

(i) for a Dixmier trace Tr_ω,

$$\mathrm{Tr}_\omega(A) = \frac{1}{d(2\pi)^d}\omega(\mathrm{Res}(A))$$

where $\mathrm{Res}(A) \in \ell_\infty/c_0$ is the residue of A;

(ii)

$$\mathrm{Tr}_\omega(A) = \frac{1}{d(2\pi)^d}\lim_{n\to\infty}(\mathrm{Res}(A))_n$$

for every Dixmier trace Tr_ω if and only if the residue $\mathrm{Res}(A)$ is convergent.

Extension of Connes' Trace Theorem

If A is a compactly supported Laplacian modulated operator, then

(i) for any normalised positive trace τ,

$$\tau(A) = \frac{1}{(2\pi)^d \log 2} B\left(\left\{\int_{\mathbb{R}^d}\int_{2^{n/d}<|s|\leq 2^{(n+1)/d}} p_A(x,s)dsdx\right\}_{n\geq 0}\right),$$

where B is the Banach limit corresponding to τ;

(ii) the equality

$$\tau(A) = \frac{1}{d(2\pi)^d}\lim_{n\to\infty}(\mathrm{Res}(A))_n$$

holds for every positive normalised trace τ on $\mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$ if and only if the residue $\mathrm{Res}(A)$ is convergent and the sequence

$$\left\{\int_{\mathbb{R}^d}\int_{2^{\frac{n}{d}}<|s|\leq 2^{\frac{n+1}{d}}} p_A(x,s)ds\,dx\right\}_{n\geq 0} \qquad (2)$$

is almost convergent to the number $\frac{1}{d}\log 2\cdot\lim_{n\to\infty}(\mathrm{Res}(A))_n$.

Discussion

Theorem

There exists a compactly supported pseudo-differential operator Q of order $-d$ such that Q is Dixmier-measurable but Q is not measurable with respect to positive normalised trace.

This example of the operator Q is interesting, because it shows how different traces of compactly supported PDOs of order $-d$ are from traces of *classical* compactly supported PDOs of order $-d$.
On the classical operators there is one trace and one Wodzicki's residue.
Even the natural vector generalisation Res of the Wodzicki's residue, whilst it does capture the behaviour of Dixmier traces on the non-classical operators, still does not capture the full behaviour of positive traces on the non-classical operators.

References

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