Introduction to non-commutative analysis and integration

Fedor Sukochev

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Measurability for all traces (which we shall not discuss much)

Recall that if $\mathcal{I}$ is an ideal in $\mathcal{B}(H)$, then a unitarily invariant linear functional $\varphi : \mathcal{I} \to \mathbb{C}$ is said to be a trace.
Since $U^{-1}TU - T = [U^{-1}, TU]$ for all $T \in \mathcal{I}$ and for all unitaries $U \in \mathcal{B}(H)$, and since the unitaries span $\mathcal{B}(H)$, it follows that traces are precisely the linear functionals on $\mathcal{I}$ satisfying the condition

$$\varphi(TS) = \varphi(ST), \quad T \in \mathcal{I}, S \in \mathcal{B}(H).$$

Measurability for all traces (which we shall not discuss much)

The latter may be reinterpreted as the vanishing of the linear functional φ on the commutator subspace $[\mathcal{J}, \mathcal{B}(H)]$. However, the commutator subspace of the ideal $\mathcal{L}_{1,\infty}$ is an ideal in $\mathcal{L}_{1,\infty}$ (as opposed to $\mathcal{B}(H)$). For $p > 1$, the ideal $\mathcal{L}_{p,\infty}$ does not admit a non-zero trace while for $p = 1$, there exists a plethora of traces on $\mathcal{L}_{1,\infty}$.
We are mostly interested in *normalised traces* $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$, that is, satisfying $\varphi(T) = 1$ whenever $0 \leq T$ is such that $\mu(k, T) = 1/(k+1)$ for all $k \geq 0$. We do not require continuity of a normalised trace.

Measurability: definitions

Definition

Let $A \in \mathcal{L}_{1,\infty}$. If there exists $c \in \mathbb{C}$ such that $\varphi(A) = c$

(a) for every normalised trace φ on $\mathcal{L}_{1,\infty}$;

(b) for every positive normalised trace φ on $\mathcal{L}_{1,\infty}$;

(c) for every Dixmier trace φ on $\mathcal{L}_{1,\infty}$,

then A is called

(a) universally measurable;

(b) measurable;

(c) Dixmier measurable.

Universal measurability: criteria

Simple lemma

Lemma

All normalised traces on $\mathcal{L}_{1,\infty}$ take the value $z \in \mathbb{C}$ on the operator T if and only if

$$T - z \cdot \mathrm{diag}\Big(\Big\{\frac{1}{k+1}\Big\}_{k\geq 0}\Big) \in [\mathcal{L}_{1,\infty}, \mathcal{L}(H)].$$

but very non-trivial theorem:

Theorem

An operator $A \in \mathcal{L}_{1,\infty}$ is universally measurable if and only if

$$\sum_{k=0}^{n} \lambda(k, A) = c \cdot \log(n+1) + O(1), \quad n \geq 0.$$

Universal measurability: comments on the literature.

The breakthrough in the describing/understanding the commutator subspace is in **N. Kalton**, *Spectral characterization of sums of commutators. I*, J. Reine Angew. Math. **504** (1998), 115–125. The statement above appeared first in N. Kalton, S. Lord, D. Potapov, F. S., *Traces on compact operators and the noncommutative residue,* Adv. Math. **235** (2013), 1–55. For a detailed proof see Theorem 5.7.6 and Theorem 10.1.3 in S. Lord, F. S., D. Zanin, *Singular Traces: Theory and Applications,* volume 46 of Studies in Mathematics. De Gruyter, 2013. For normal operators, the assertion is firstly proved in K. Dykema, T. Figiel, G. Weiss, M. Wodzicki, *Commutator structure of operator ideals,* Adv. Math. **185** (2004), no. 1, 1–79. Substantial applications of this result are to be found in *Universal measurability and the Hochschild class of the Chern character*, J. Spectr. Theory 6 (2016), 1–41 due to A. Carey, A. Rennie, F. S., and D. Zanin

Measurability: criteria

Theorem

An operator $A \in \mathcal{L}_{1,\infty}$ is measurable if and only if the sequence

$$\Big\{ \sum_{k=2^n-1}^{2^{n+1}-2} \lambda(k, A) \Big\}_{n \geq 0}$$

is almost convergent (i.e. all Banach limits take the same value on it).

Proof.

The assertion follows from "Pietsch theorem" discussed in the preceding lecture. □

Dixmier measurability: criteria

Theorem

An operator $A \in \mathcal{L}_{1,\infty}$ is Dixmier measurable if and only if

$$\sum_{k=0}^{n} \lambda(k, A) = c \cdot \log(n) + o(\log(n)), \quad n \to \infty.$$

Proof.

we have that

$$\mathrm{tr}_\omega(A) = \omega\Big(\Big\{\frac{1}{\log(n+2)} \sum_{k=0}^{n} \lambda(k, A)\Big\}_{n\geq 0}\Big), \quad A \in \mathcal{L}_{1,\infty},$$

for every singular state ω. If $x \in l_\infty$, then all singular states take the same value on x if and only if x is convergent. This completes the proof. □

$\zeta-$function

Definition

For a positive compact operator A, the mapping

$$s \to \mathrm{Tr}(A^s)$$

is called $\zeta-$function of the operator A.

If $A \in \mathcal{L}_{1,\infty}$, then $\zeta-$function is defined for $\Re(s) > 1$. Indeed, let $x = \mathrm{diag}\{\frac{1}{k+1}\}_{k\geq 0}$. We have

$$|\mathrm{Tr}(A^s)| \leq \mathrm{Tr}(A^{\Re(s)}) \leq \|A\|_{1,\infty}^{\Re(s)} \sum_{k\geq 1} k^{-\Re(s)} = \|A\|_{1,\infty}^{\Re(s)} \zeta(\Re(s)).$$

Moreover, since Riemann $\zeta-$function has a simple pole at $s = 1$, it is bounded by $(s-1)^{-1}$ as $s \to 1$, then so is $\mathrm{Tr}(A^{1+s})$.

Dixmier measurability criteria in terms of $\zeta-$function

Theorem

An operator $0 \leq A \in \mathcal{L}_{1,\infty}$ is Dixmier measurable if and only if there exists a limit

$$\lim_{s \downarrow 1}(s-1)\mathrm{Tr}(A^s).$$

The remaining part of the lecture is devoted to the proof of this result.

Extended residues

Let γ be a singular state on the algebra $L_\infty(0,1)$ supported at 0 (that is, it vanishes at each function supported outside some neighbourhood of 0). Consider the functional ζ_γ defined on the positive cone of $\mathcal{L}_{1,\infty}$ by setting

$$\zeta_\gamma(A) = \gamma\Big(s \to s\mathrm{Tr}(A^{1+s})\Big), \quad 0 \le A \in \mathcal{L}_{1,\infty}.$$

Using the inequalities

$$\mathrm{Tr}(A_1^{1+s} + A_2^{1+s}) \le \mathrm{Tr}((A_1 + A_2)^{1+s}) \le 2^s\mathrm{Tr}(A_1^{1+s} + A_2^{1+s}),$$

we obtain that ζ_γ is additive (and homogeneous) on the positive cone of $\mathcal{L}_{1,\infty}$. Hence, it extends to a linear functional (automatically, a trace) on the whole $\mathcal{L}_{1,\infty}$.

Proof of the "only if" part

If $A_1, A_1 \in \mathcal{L}_{1,\infty}$ are positive elements such that $A_2 \prec\prec A_1$, then

$$\mathrm{Tr}(A_2^{1+s}) \leq \mathrm{Tr}(A_1^{1+s}), \quad s \in (0,1).$$

Thus, $\zeta_\gamma(A_2) \leq \zeta_\gamma(A_1)$.
In other words, ζ_γ is monotone with respect to a Hardy-Littlewood pre-order. By the abstract description of Dixmier traces, it follows that every ζ_γ is a Dixmier trace.
If $0 \leq A \in \mathcal{L}_{1,\infty}$ is such that $\mathrm{tr}_\omega(A) = c$ for every ω, then $\zeta_\gamma(A) = c$ for every γ.
If $x \in L_\infty(0,1)$ is such that $\gamma(x) = c$ for every singular state γ supported at 0, then there exists a limit $x(+0)$. Applying this to

$$x : s \to s\mathrm{Tr}(A^{1+s}), \quad s \in (0,1),$$

we complete the proof of the "only if" part.

Karamata theorem

Theorem

Let $\beta : (0, \infty) \to (0, \infty)$ *be an increasing function such that* $\beta(t) \leq t$ *for all* $t > 0$*. Let*

$$h(s) = \int_0^\infty e^{-us} d\beta(u), \quad s > 0.$$

We have

$$\lim_{s \to 0} s \cdot h(s) = \lim_{t \to \infty} \frac{\beta(t)}{t}$$

provided that the limit in the left hand side exists.

Proof of the “if” part. I

For $0 \le A \in \mathcal{L}_{1,\infty}$ (we assume for simplicity that $\|A\|_\infty \le 1$) set

$$\beta(t) = -\int_0^t e^{-v} d\mathrm{Tr}(E_A(e^{-v},\infty)).$$

It is immediate that

$$h(s) = -\int_0^\infty e^{-us} \cdot e^{-u} d\mathrm{Tr}(E_A(e^{-u},\infty)) =$$

$$= -\int_0^1 \lambda^{1+s} d\mathrm{Tr}(E_A(\lambda,\infty)) = \mathrm{Tr}(A^{1+s}).$$

If $\lim_{s\to 0} s\mathrm{Tr}(A^{1+s}) = c$, then

$$c = \lim_{s\to 0} s \cdot h(s) = \lim_{t\to\infty} \frac{\beta(t)}{t}.$$

Proof of the "if" part. II

Note that

$$\beta(t) = -\int_{e^{-t}}^{1} \lambda d\mathrm{Tr}(E_A(\lambda,\infty)) =$$

$$= -\int_{e^{-t}}^{\infty} \lambda d\mathrm{Tr}(E_A(\lambda,\infty)) = \mathrm{Tr}(AE_A(e^{-t},\infty)).$$

Since $0 \le A \in \mathcal{L}_{1,\infty}$, it follows that

$$\mathrm{Tr}(AE_A(e^{-t},\infty)) = \int_0^{e^t} \mu(s,A)ds + O(1).$$

Hence, there exists a limit

$$\lim_{t\to\infty} \frac{1}{t}\int_0^{e^t} \mu(s,A)ds = c.$$

Applying Dixmier measurability criterion, we complete the proof of the "if" part.