Introduction to non-commutative analysis and integration

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Characters of l_∞

Definition

Bounded linear functional $f : l_\infty \to \mathbb{C}$ is called character if

$$f(xy) = f(x)f(y), \quad x, y \in l_\infty.$$

If $n \geq 0$, then the mapping $x \to x(n)$ is a character (we call them normal). Our aim is to show that there are **other** characters (we call them singular).

Characters are states

Let f be a character and let $A \subset \mathbb{Z}_+$. We have

$$f(\chi_A) = f(\chi_A \cdot \chi_A) = f(\chi_A) \cdot f(\chi_A).$$

Thus, either $f(\chi_A) = 0$ or $f(\chi_A) = 1$.
In particular, either $f(1) = 1$ or $f(1) = 0$. If $f(1) = 0$, then

$$f(x) = f(x \cdot 1) = f(x) \cdot f(1) = 0, \quad x \in l_\infty.$$

Thus, $f(1) = 1$ unless $f = 0$.
If $x \in l_\infty$ is positive, then there exists a sequence $\{x_n\}_{n \geq 0} \subset l_\infty$ such that each x_n is positive, takes only finitely many values and such that $x_n \to x$ in l_∞. Since f is bounded, it follows that $f(x_n) \to f(x)$. Each x_n is a linear combination of indicator functions (with positive coefficients). Thus, $f(x_n) \geq 0$ for every $n \geq 0$. Hence, $f(x) \geq 0$.

Characters are states

Let f be a character. For every $n \geq 0$, we have that $f(\chi_{\{n\}}) = 0$ or $f(\chi_{\{n\}}) = 1$.
Suppose that $f(\chi_{\{n\}}) = 1$ for some $n \geq 0$. Let $A = \mathbb{Z}_+ \backslash \{n\}$. We have $f(\chi_A) = f(1) - f(\chi_{\{n\}}) = 0$. Let $y \in l_\infty$ be a positive element supported on A. We have $0 \leq y \leq \|y\|_\infty \chi_A$. Since f is positive, it follows that $0 \leq f(y) \leq \|y\|_\infty f(\chi_A) = 0$. Thus, $f(y) = 0$. It follows that $f(x) = f(x\chi_{\{n\}}) + f(x\chi_A) = x(n), \quad x \in l_\infty$.
Suppose that $f(\chi_{\{n\}}) = 0$ for every $n \geq 0$. We claim that f is a singular state. Since f is already a state, it suffices to verify that f vanishes on c_0. For this purpose, fix $x \in c_0$ and let

$$y_n = x\chi_{[0,n]}, \quad z_n = x\chi_{(n,\infty)}.$$

Since $x \in c_0$, it follows that $z_n \to 0$ in l_∞ as $n \to \infty$. Taking into account that f is bounded, we conclude that also $f(z_n) \to 0$ as $n \to \infty$. On the other hand, $f(y_n) = 0$ for every $n \geq 0$ by the assumption. Thus, $f(x) \to 0$ as $n \to \infty$. Since there is no dependence on n, it follows that $f(x) = 0$.

Characters are extreme points

Let f be a character. We claim it is an extreme point for the set of all states. Let $f = \frac{1}{2}(g+h)$, where g and h are states on l_∞. Take $A \subset \mathbb{Z}_+$. We have either $f(\chi_A) = 0$ or $f(\chi_A) = 1$.
If $f(\chi_A) = 0$, then $g(\chi_A) + h(\chi_A) = 0$. Since $g, h \geq 0$, it follows that $g(\chi_A) = h(\chi_A) = 0$.
If $f(\chi_A) = 1$, then $f(\chi_{A^c}) = 0$. Hence, $g(\chi_{A^c}) = h(\chi_{A^c}) = 0$. Thus, $g(\chi_A) = h(\chi_A) = 1$.
Thus, for every $A \subset \mathbb{Z}_+$, we have that $g(\chi_A) = h(\chi_A) = f(\chi_A)$. Thus, $g(x) = h(x) = f(x)$ for every $x \in l_\infty$ which takes only finitely many values. By continuity, $g(x) = h(x) = f(x)$ for every $x \in l_\infty$. Thus, $g = h = f$.

Extreme points are characters

Let f be an extreme points for the set of states (or for the set of all singular states). Fix an element $0 \leq y \leq 1$ in l_∞. Define bounded functionals f_1 and f_2 on l_∞ by setting

$$f_1(x) = f(x) + f(xy) - f(x)f(y), \quad x \in l_\infty,$$

$$f_2(x) = f(x) - f(xy) + f(x)f(y), \quad x \in l_\infty.$$

If $x \geq 0$, then

$$f_1(x) = f(x) \cdot (1 - f(y)) + f(xy) \geq 0$$

and, similarly, $f_2(x) \geq 0$. Thus, f_1 and f_2 are states. If f is singular, then so are f_1 and f_2.

Clearly, $f = \frac{1}{2}(f_1 + f_2)$. Since f is an extreme point, it follows that $f_1 = f_2$. Thus,

$$f(xy) = f(x)f(y), \quad x \in l_\infty.$$

Since $0 \leq y \leq 1$ in l_∞ is arbitrary, it follows that f is a character.

Singular characters do exist

States form a convex compact in weak* topology. So do singular states. By Krein-Milman theorem, every singular state can be approximated by a convex combination of the extreme points. In particular, extreme points for the set of singular states do exist.

Taking f to be an extreme point for the set of singular states, we conclude the existence of singular characters.

In fact, there are lots of them — the cardinality of the set of all characters is $2^{2^{\mathbb{N}}}$.

Citing classics

Ola Bratteli and Derek W. Robinson **Operator Algebras and Quantum Statistical Mechanics 1**
Corollary 2.3.21 and Proposition 2.3.27. *Let ω be a state over an abelian C^*-algebra $\mathcal{U}$. It follows that ω is a pure state if and only if $\omega(AB) = \omega(A)\omega(B)$ for all $A, B \in \mathcal{U}$. In other words, a non-zero linear functional on $\mathcal{U}$ is a pure state if and only if it is a character.*

Filters

Definition

A set $\mathcal{F}$ of subsets of the set X is said to be a filter on X if the following conditions hold

(i) If the set $A \subset X$ contains some set from $\mathcal{F}$, then $A \in \mathcal{F}$;

(ii) The intersection of every finite family of sets from $\mathcal{F}$ belongs to $\mathcal{F}$;

(iii) $\mathcal{F}$ does not contain an empty set.

There are two important examples of filters:

- the set of all neighbourhoods of a non-empty set (a point, for example) in a topological space is a filter. Such a filter is called a principal filter;
- if X is an infinite set, then the complements of all finite sets form a filter in X. Such a filter is called a Fréchet filter.

Ultrafilters

The following definition introduces the concept of a maximal element in the set of all filters.

Definition

A filter $\mathcal{F}$ on the set X is said to be an ultrafilter if there is no filter $\mathcal{F}'$ on X, such that $\mathcal{F} \subsetneq \mathcal{F}'$.

Any principal filter on a non-empty set X is an ultrafilter on X.

Proposition

A filter $\mathcal{F}$ on the set X is an ultrafilter if and only if for every $A \subset X$ we have that either A or its complement belongs to $\mathcal{F}$.

It follows directly from the latter proposition that Fréchet filter on an infinite set X is not an utrafilter. Indeed, if an infinite set $A \subset X$ is such that its complement is also infinite, then neither A nor its complement belongs to a Fréchet filter. Hence, Fréchet filter is not an utrafilter by the later proposition.

Free filters

Definition

A filter is said to be free if the intersection of all its elements is empty.

For example, the Fréchet filter is free.
Note that an ultrafilter on an infinite set X is free if and only if it contains the Fréchet filter on X. Indeed, since a free ultrafilter contains no finite set, it follows that it contains all cofinite subsets of X, which is exactly the Fréchet filter.

Limits along filters

Definition

Let f be a mapping from the set X to a topological space Y and let $\mathcal{F}$ be a filter on the set X. An element $y \in Y$ is said to be a limit of f along the filter $\mathcal{F}$ (we write $y = \lim_{\mathcal{F}} f$) if $f^{-1}(V) \in \mathcal{F}$ for every neighbourhood V of y.

The usual limit functional may be considered as a limit along the Fréchet filter.

Limits along ultrafilters

Proposition

The limit along any free utrafilter on $\mathbb{N}$ is a state on l_∞.

Proof.
Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Let us show that the functional $l : l_\infty \to \mathbb{R}$, given by the following formula

$$l(x) := \lim_{\mathcal{U}} x, \ \ x \in l_\infty$$

is well-defined. **skipped**

Proof cntd

The positivity of l easily follows from the definition of a limit along the filter. Indeed, for every $x \geq 0$ and $a < 0$ there exists $\varepsilon = a/2$ such that the set $\{n : |x_n - a| < \varepsilon\}$ is empty and, thus does not belong to an ultrafilter $\mathcal{U}$. Hence, a can not be a limit of x along $\mathcal{U}$.
Since every free ultrafilter contains the Fréchet filter and since the limit along the Fréchet filter is the usual limit (if exists), it follows that $l(x) = \lim_{n\to\infty} x_n$ for every convergent sequence x.

Note that in general the latter proposition does not hold for free filters (since the functional l is not well-defined) and for non-free ultrafilters (since, the limit along non-free ultrafilter does not extend the usual limit).

Characters vs ultrafilters

There exists a canonical bijection between the set of all (singular) characters and the set of all (free) ultrafilters. Every character is a limit along some ultrafilter. Conversely, every limit along an ultrafilter is a character. Let f be a character on l_∞. Then for every $A \subset Z_+$ we have $f(\chi_A) = 1$ or $f(\chi_A) = 0$. Let U be a subset of such A for which $f(\chi_A) = 1$. Let us show that U is an ultrafilter.
It suffices to verify the following conditions 1-4 (see e.g. Wiki/ultrafilerts/ and section ultrafilter on the powerset of a set).
1) If $\chi_\emptyset = 0$, then $f(\chi_\emptyset) = f(0) = 0$, that is $\emptyset \notin U$.
2) If $A \subset B$ and $A \in U$, then $B \in U$. Indeed, $f(\chi_A) = 1$, we need to show that $f(\chi_B) = 1$. By a contradiction, assume that $f(\chi_B) = 0$. We have $\chi_A = \chi_A \cdot \chi_B$. Hence $1 = f(\chi_A) = f(\chi_A) \cdot f(\chi_B) = 1 \cdot 0 = 0$.

(Singular) Characters vs ultrafilters-2

3) Let $A, B \in U$, let us show that $A \cap B \in U$. In other words that $f(\chi_A) = 1$ and $f(\chi_B) = 1$ imply $f(\chi_{A\cap B}) = 1$. By a contradiction, assume that $f(\chi_{A\cap B}) = 0$. Denote $C = A \backslash B$ and $D = B \backslash A$. We have $\chi_C = \chi_A - \chi_{A\cap B}$ and therefore $f(\chi_C) = f(\chi_A) - f(\chi_{A\cap B}) = 1 - 0 = 1$. Similarly, $f(\chi_D) = 1$. A contradiction.
4) Let $A \subset Z_+$ and $B = Z_+ \backslash A$; we have to show that either $A \in U$ or else $B \in U$. By a contradiction, assume that $f(\chi_A) = f(\chi_B) = 0$. Then $1 = f(1) = f(\chi_A) + f(\chi_B) = 0 + 0 = 0$, etc.

(Singular) Characters vs ultrafilters-3

In what follows, we use a notation

$$\lim_{n\to\omega} x(n)$$

for a limit of the sequence $\{x(n)\}_{n\geq 0}$ along an ultrafilter ω.
Definition: $\lim_{n\to\omega} x_n = a$ if for every $\epsilon > 0$ the set$\{n : a-\epsilon < x_n < a+\epsilon\}$ belongs to ω.
Let f and ω be as on the preceding slide. We claim that $f = \lim_{n\to\omega}$.

Characters vs ultrafilters-4

Fix $x \in l_\infty$, and set $a = f(x)$. Fix $\epsilon > 0$ and consider three sets:
$A_1 = \{n : a - \epsilon < x_n < a + \epsilon\}$, $A_2 = \{n : x_n \geq a + \epsilon\}$ and
$A_3 = \{n : a - \epsilon \geq x_n\}$.
If $f(\chi_{A_2}) = 1$, then

$$f(x) - a = f(x - a) = f(x - a)f(\chi_{A_2}) = f((x - a)\chi_{A_2}) \geq f(\epsilon\chi_{A_2}) = \epsilon,$$

that is $f(x) > a + \epsilon$. However $f(x) = a$, a contradiction. Hence, $f(\chi_{A_2}) = 0$ and similarly $f(\chi_{A_3}) = 0$. Thus, $f(\chi_{A_1}) = 1$. By the definition of ω we obtain $A_1 \in \omega$.
Due to the arbitrariness of ϵ, we arrive to $\lim_{n\to\omega} x_n = a = f(x)$. Due to the arbitrariness of $x \in l_\infty$, we arrive at $f(x) = \lim_{n\to\omega} x_n$.
Thus, every character is a limit along a free ultrafilter ω.

Limits along ultrafilters are characters

Proposition

The limit along any free utrafilter on $\mathbb{N}$ is a character on l_∞.

Proof.

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and let $x, y \in l_\infty$. Denote $a := \lim_{\mathcal{U}} x$ and $b := \lim_{\mathcal{U}} y$. Hence, by definition of limits along ultrafilter, for every $\varepsilon_1, \varepsilon_2 > 0$ sets $\{n : |x_n - a| \leq \varepsilon_1\}$ and $\{n : |y_n - b| \leq \varepsilon_2\}$ belong to $\mathcal{U}$.
Fix $\varepsilon > 0$ and set $\varepsilon_1 = \frac{\varepsilon}{2\|y\|}$ and $\varepsilon_2 = \frac{\varepsilon}{2\|x\|}$. Since $|x_n y_n - ab| \leq \|y\|_{l_\infty} |x_n - a| + \|x\|_{l_\infty} |y_n - b|$, it follows that

$$\{n : |x_n - a| \leq \varepsilon_1\} \cap \{n : |y_n - b| \leq \varepsilon_2\} \subset \{n : |x_n y_n - ab| \leq \varepsilon\} \in \mathcal{U},$$

by definition of a filter (parts (i) and (ii)).
Consequently, $\lim_{\mathcal{U}} xy = \lim_{\mathcal{U}} x \lim_{\mathcal{U}} y$ by definition of limits along ultrafilter.

Characters are not translation invariant nor Cesaro nor dilation invariant

Characters are not dilation invariant states. Indeed, let $A = \cup_{n\geq 0}[2^{2n}, 2^{2n+1})$. Then $A \cap 2A = \emptyset$ and $A \cup 2A = \mathbb{N}$. If $f = f \circ \sigma_2$, then $f(\chi_A) = f(\chi_{2A})$ since $\chi_A + \chi_{2A} = 1 - e_0$, hence $f(\chi_A) = 1/2$. However, we should have 0 or 1.

Characters are not Cesaro invariant ($f \neq \phi \circ C$). Hence, it is translation invariant. Set $A = \cup_{n\geq 0}\{2n\}$. Then $A \cap T(A) = \emptyset$ and $A \cup (T(A)) = \mathbb{N}$. Thus $f(\chi_A) = 1/2$, but we should have 0 or 1. Thus, characters are not translation invariant and not Cesaro invariant.

Dixmier and Pietsch traces

Dixmier: Let ω be a singular state on l_∞. Define a functional tr_ω on the positive cone of $\mathcal{L}_{1,\infty}$ by setting

$$\mathrm{tr}_\omega(A) = \omega\Big(\Big\{\frac{1}{\log(n+2)}\sum_{k=0}^{n}\mu(k,A)\Big\}_{n\geq 0}\Big), \quad 0 \leq A \in \mathcal{L}_{1,\infty}.$$

Pietsch: For every (positive) shift-invariant linear functional θ on ℓ_∞ the functional

$$\tau(A) = \theta\left(\left\{\sum_{k=2^n-1}^{2^{n+1}-2}\lambda(k,A)\right\}_{n\geq 0}\right),\ A \in \mathcal{L}_{1,\infty}$$

extend by linearity to a (positive) trace on $\mathcal{L}_{1,\infty}$.

Extreme points

1) The mapping $\omega \to \mathrm{tr}_\omega$ is an affine mapping from the set of all states on ℓ_∞ onto the set of all Dixmier traces on $\mathcal{L}_{1,\infty}$. Does it motivate the study of Dixmier traces generated by characters (that is generated by extreme points of the set of all states on ℓ_∞)?
2) The mapping $\omega \to \tau_\omega$ is an affine and isometrical mapping from the set of all (positive) translation-invariant functionals (=Banach limits) on ℓ_∞ onto the set of all (positive) traces $\mathcal{L}_{1,\infty}$. Does it motivate the study of (positive) traces generated by by extreme points of the set of Banach limits on ℓ_∞?