Introduction to non-commutative analysis and integration

Fedor Sukochev

July 15, 2017

Why not Dixmier approach?

Let i be a mapping from all extended limits to all positive normalised traces, given by $i : \omega \to \mathrm{tr}_\omega$.
There are 2 fundamental faults in Dixmier's approach to traces on $\mathcal{L}_{1,\infty}$:

1. The mapping i is not injective.
2. The mapping i is not surjective.

Explanation:

1. It was shown in Theorem 40[1] that there are at least two distinct ω's which produce the same trace tr_ω.
2. There are (positive, normalised) traces on $\mathcal{L}_{1,\infty}$ which fail to be Dixmier traces[2].

[1] F. S., A. Usachev, D. Zanin, Generalized limits with additional invariance properties and their applications to noncommutative geometry, Adv. Math., **239** (2013)

[2] N. Kalton, F. S., Rearrangement-invariant functionals with applications to traces on symmetrically normed ideals, Canad. Math. Bull. 51(1) (2008)

Program

Our aim here is to present an approach due to Pietsch which allows to describe traces on $\mathcal{L}_{1,\infty}$ in terms of so-called Banach limits. The correspondence $\omega \to \varphi_\omega$ in this approach is both injective and surjective. The definition enjoys the same level of simplicity and constructivism as that of Dixmier traces. The original Pietsch's approach[3] was greatly simplified in SSUZ[4].

[3] A. Pietsch, Traces on operator ideals and related linear forms on sequence ideals (part I), Indag. Math. (N.S.) **25** (2014), no. 2, 341–365.

[4] E.Semenov, F.S., A.Usachev, D.Zanin, Traces on $\mathcal{L}_{1,\infty}$ and Banach limits, Adv. Math., **285** (2015), 568–628.

New construction of traces

For every shift-invariant linear functional θ on ℓ_∞ the functional

$$\tau(A) = \theta\left(\left\{\sum_{k=2^n-1}^{2^{n+1}-2} \lambda(k,A)\right\}_{n\geq 0}\right),\ A \in \mathcal{L}_{1,\infty}$$

extends by linearity to a trace on $\mathcal{L}_{1,\infty}$.

Advantages:

1. Allows to construct ALL traces on $\mathcal{L}_{1,\infty}$;

   There are singular traces different from Dixmier traces [5].

2. Establishes a bijective correspondence between shift-invariant functionals on l_∞ and traces on $\mathcal{L}_{1,\infty}$.

[5] N. Kalton, F. S., Symmetric norms and spaces of operators, J. Reine Angew. Math. 621 (2008), 81–121.

Banach limits: definition

Let $T : l_\infty \to l_\infty$ be a translation operator. That is,

$$T : (x(0), x(1), x(2), \cdots) \to (x(1), x(2), \cdots).$$

Definition

State ω on the algebra l_∞ is a Banach limit if $\omega = \omega \circ T$.

Banach limits were introduced by Banach and thoroughly investigated by Lorentz. Their existence is usually demonstrated via Hahn-Banach extension theorem (see e.g. end of Chapter II[6]). We present a simple proof of existence of the Banach limits (due to Mazur, Raimi, Connes).
The word "limit" is due to the fact that every Banach limit is an extension of the limit functional (hence, in a certain sense, it is a limiting procedure).

[6] S. Banach, Théorie des opérations linéaires, 1993, reprint of the 1932 original.

Banach limits: existence

Let $\omega : l_\infty \to \mathbb{C}$ be a singular state and let $C : l_\infty \to l_\infty$ be a Cesaro operator given by the formula

$$(Cx)(n) = \frac{1}{n+1}\sum_{k=0}^{n} x(k), \quad n \geq 0.$$

We claim that $\omega \circ C$ is a Banach limit.
Indeed, for every $x \in l_\infty$, we have

$$(CTx - Cx)(n) = \frac{1}{n+1}(x(n+1) - x(0)) = o(1), \quad n \to \infty.$$

In other words, $CTx - Cx \in c_0$. Since ω vanishes on c_0, it follows that

$$((\omega\circ C)\circ T)(x) = \omega(CTx) = \omega(Cx)+\omega(CTX - Cx) = \omega(Cx) = (\omega\circ C)(x).$$

Since $x \in l_\infty$ is arbitrary, it follows that $(\omega \circ C) \circ T = \omega \circ C$. That is, $\omega \circ C$ is a Banach limit.
Note that not all Banach limits are of this form.

Pietsch Theorem (as given in SSUZ)

Theorem

For every Banach limit ω, the mapping $\varphi_\omega : \mathcal{L}_{1,\infty} \to \mathbb{C}$ given by the formula

$$\varphi_\omega(A) = \frac{1}{\log(2)} \cdot \omega\Big(\Big\{ \sum_{k=2^n-1}^{2^{n+1}-2} \lambda(k, A) \Big\}_{n\geq 0}\Big), \quad A \in \mathcal{L}_{1,\infty},$$

is a positive normalised trace on $\mathcal{L}_{1,\infty}$. Conversely, every positive normalised trace on $\mathcal{L}_{1,\infty}$ is of the form φ_ω for some (unique) Banach limit ω.

The proof of this theorem is not terribly complicated, but is quite long. We do not prove this result here.

The connection between Pietsch and Dixmier approaches.

Let ω be a state and, so, $\omega \circ C$ be a Banach limit. Consider the trace $\varphi_{\omega\circ C}$.
It is immediate that

$$C\Big(\Big\{\sum_{k=2^n-1}^{2^{n+1}-2}\lambda(k,A)\Big\}_{n\geq 0}\Big)=\Big\{\frac{1}{n+1}\sum_{k=0}^{2^{n+1}-2}\lambda(k,A)\Big\}_{n\geq 0}.$$

Thus,

$$\varphi_{\omega\circ C}(A)=\frac{1}{\log(2)}\cdot\omega\Big(\Big\{\frac{1}{n+1}\sum_{k=0}^{2^{n+1}-2}\lambda(k,A)\Big\}_{n\geq 0}\Big),\quad A\in\mathcal{L}_{1,\infty}.$$

By the abstract description of Dixmier traces, we have that $\varphi_{\omega\circ C}$ is a Dixmier trace. Conversely, using the argument in KSS or SS, one can show that every Dixmier trace is of the form $\varphi_{\omega\circ C}$.

Existence of non-Dixmier traces

Theorem

There are positive normalised traces on $\mathcal{L}_{1,\infty}$ which are not Dixmier traces.

In subsequent slides, we sketch the proof of this result.

Proofs. I

For a given $n \in \mathbb{N}$, let $A_n \in \mathcal{L}_{1,\infty}$ be a positive operator such that

$$\mu(A_n) = \sup_{\substack{i,j \in \mathbb{N} \\ j \leq i}} 2^{-j-n(i+2)^2} \chi_{[0, 2^{j+n(i+2)^2})}.$$

A direct verification shows that $\|A_n\|_{1,\infty} \leq 1$ and that

$$A_n \prec\prec \frac{2}{n} \cdot \left\{ \frac{1}{k+1} \right\}_{k \geq 0}.$$

By the definition of Dixmier trace, we have that

$$\mathrm{tr}_\omega(A_n) \leq \frac{2}{n}, \quad n \in \mathbb{N},$$

for every singular state ω.

Proofs. II

Let $\mathfrak{B}$ be the collection of all Banach limits. Sucheston proved that

$$\sup_{\omega\in\mathfrak{B}}\omega(x)=\lim_{k\to\infty}\sup_{m\geq 0}\frac{1}{k}\sum_{i=m}^{k+m-1}x(i),\quad x\in l_\infty.$$

Let $\mathcal{PT}$ be the collection of all positive normalised traces on $\mathcal{L}_{1,\infty}$. We have

$$\sup_{\varphi\in\mathcal{PT}}\varphi(A)=\frac{1}{\log(2)}\cdot\lim_{k\to\infty}\sup_{m\geq 0}\frac{1}{k}\sum_{i=2^m-1}^{2^{k+m}-2}\lambda(i,A),\quad A=A^*\in\mathcal{L}_{1,\infty}.$$

Proofs. III

Let $A = A_n$ and, for a given $k \in \mathbb{N}$, let $m = n(k+2)^2$. We have

$$\sup_{\varphi \in \mathcal{PT}} \varphi(A_n) = \frac{1}{\log(2)} \cdot \liminf_{k \to \infty} \frac{1}{k} \sum_{i=2^{n(k+2)^2}-1}^{2^{k+n(k+2)^2}-2} \mu(i, A), \quad A = A^* \in \mathcal{L}_{1,\infty}.$$

A direct computation shows that

$$\sum_{i=2^{n(k+2)^2}-1}^{2^{k+n(k+2)^2}-2} \mu(i, A) = \frac{k}{2}.$$

Thus,

$$\sup_{\varphi \in \mathcal{PT}} \varphi(A_n) \geq \frac{1}{2\log(2)} > \frac{2}{n} \geq \sup_{\omega} \mathrm{tr}_\omega(A_n)$$

for every $n \geq 3$. This completes the proof.

How many traces are there?

Theorem

There exist $2^{2^{\mathbb{N}}}$ positive normalised traces on $\mathcal{L}_{1,\infty}$.

Proof.

Indeed, we have a bijection between positive normalised traces and Banach limits. Also, Pietsch constructed a bijection between Banach limits and ultrafilters. It is well known that there are exactly $2^{2^{\mathbb{N}}}$ ultrafilters. □