Introduction to non-commutative analysis and integration

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General notations

Fix throughout a separable infinite dimensional Hilbert space H. We let $B(H)$ denote the algebra of all bounded operators on H. For a compact operator T on H, let $\mu(k, T)$ denote $k-$th largest singular value (these are the eigenvalues of $|T|$). The sequence $\mu(T) = \{\mu(k, T)\}_{k\geq 0}$ is referred to as to the singular value sequence of the operator T. The standard trace on $B(H)$ is denoted by Tr.

Fix an orthonormal basis in H (the particular choice of a basis is inessential). We identify the algebra l_∞ of bounded sequences with the subalgebra of all diagonal operators with respect to the chosen basis. For a given sequence $\alpha \in l_\infty$, we denote the corresponding diagonal operator by $\mathrm{diag}(\alpha)$.

Principal ideals $\mathcal{L}_{p,\infty}$

Let $\mathcal{L}_{p,\infty}$ be the principal ideal in $B(H)$ generated by the element $A_0 = \mathrm{diag}(\{(k+1)^{-\frac{1}{p}}\}_{k\geq 0})$. Equivalently,

$$\mathcal{L}_{p,\infty} = \{A : \sup_{k\geq 0}(k+1)^{\frac{1}{p}}\mu(k,A) < \infty\}.$$

In Noncommutative Geometry, a compact operator A is called an infinitesimal of order $\frac{1}{p}$ if

$$\mu(k,A) = O((k+1)^{-\frac{1}{p}}), \quad k \in \mathbb{Z}_+.$$

In other words, $\mathcal{L}_{p,\infty}$ is the set of all infinitesimals of order $\frac{1}{p}$.

Traces on $\mathcal{L}_{1,\infty}$

Definition

A linear functional $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is called trace if $\varphi(AB) = \varphi(BA)$ for every $A \in \mathcal{L}_{1,\infty}$ and for every $B \in B(H)$.

Equivalently, a linear functional is a trace if it is unitarily invariant:

$$\varphi(U^{-1}AU) = \varphi(A), \quad A \in \mathcal{L}_{1,\infty}, \quad U \in B(H), \quad U^*U = UU^* = 1.$$

There exists a plethora of traces on $\mathcal{L}_{1,\infty}$. The most famous ones are Dixmier traces (to be constructed further in this lecture).

States

Definition

Positive linear functional $\omega : l_\infty \to \mathbb{C}$ is called state if $\omega(1) = 1$.

Each state is automatically bounded functional.
If a state is order-continuous, then it is called normal. All normal states are given by the formulae

$$x \to \sum_{k \geq 0} x(k)a(k), \quad x \in l_\infty,$$

for some positive $a \in l_1$ such that $\|a\|_1 = 1$. These normal states were used in naive construction of Quantum Theory.
Algebra l_∞ is a non-separable Banach space and so is its dual $(l_\infty)^*$. Since l_1 is separable, it follows that there exist non-normal states. In fact, there are $2^{2^{\mathbb{N}}}$ states[1] and only $2^{\mathbb{N}}$ normal states.
<u>In this lecture, we deal with states</u> which are not normal.

[1] M. Nakamura, S. Kakutani, Banach limits and the Čech compactification of a countable discrete set, Proc. Imp. Acad. Tokyo, **19** (1943), 224–229.

Singular states

Let $c_0 \subset l_\infty$ be the collection of all sequences converging to 0.

Definition

A state ω on l_∞ is called singular if it vanishes on c_0.

Let us show that singular states do exist. Consider a subspace $\mathbb{C} + c_0 \subset l_\infty$. Every element x in this subspace is *uniquely* written as $x = x_1 + x_2$, where $x_1 \in \mathbb{C}$ and $x_2 \in c_0$. Set

$$\omega(x) = x_1, \quad x \in \mathbb{C} + c_0 \subset l_\infty.$$

It is obvious that $\omega(x) \leq \limsup(x)$ for every $x \in \mathbb{C} + c_0 \subset l_\infty$. By the Hahn-Banach theorem, there exists an extension of ω to l_∞ such that $\omega(x) \leq \limsup(x)$ for every $x \in l_\infty$. Applying the inequality to $-x$, we obtain that $\omega(x) \geq \liminf(x)$ for every $x \in l_\infty$. In particular, ω is positive. By definition, $\omega(1) = 1$ and, so, ω is a state. By definition, ω vanishes on c_0 and is, therefore, singular.

Dixmier traces: definition. I

Definition

Let ω be a singular state on l_∞. Define a functional tr_ω on the positive cone of $\mathcal{L}_{1,\infty}$ by setting

$$\mathrm{tr}_\omega(A) = \omega\Big(\Big\{\frac{1}{\log(n+2)}\sum_{k=0}^{n}\mu(k,A)\Big\}_{n\geq 0}\Big), \quad 0 \leq A \in \mathcal{L}_{1,\infty}.$$

This functional is *a priori* non-linear. Nevertheless, our task is to show how to extend tr_ω to be a linear functional.

Dixmier traces: definition. II

Theorem

For every singular state ω, the functional tr_ω is additive and homogeneous on the positive cone of $\mathcal{L}_{1,\infty}$.

Every additive and homogeneous functional on the positive cone of $\mathcal{L}_{1,\infty}$ admits a unique *linear* extension to the whole $\mathcal{L}_{1,\infty}$.

Definition

The extension of tr_ω to $\mathcal{L}_{1,\infty}$ by linearity is unitarily invariant (and is, therefore, a trace). We call tr_ω a Dixmier trace.

Our further aim is to prove the above Theorem.

Proofs. I

We use the following inequalities without proof. Interested reader is referred to p.85 in [LSZ].

Theorem

If A_1 and A_2 are positive compact operators, then

1. $$\sum_{k=0}^{n} \mu(k, A_1 + A_2) \leq \sum_{k=0}^{n} \mu(k, A_1) + \mu(k, A_2).$$

2. $$\sum_{k=0}^{n} \mu(k, A_1) + \mu(k, A_2) \leq \sum_{k=0}^{2n+1} \mu(k, A_1 + A_2).$$

Proofs. II

By the Theorem on previous slide, we have

$$0 \leq \sum_{k=0}^{n} \mu(k, A_1) + \mu(k, A_2) - \mu(k, A_1 + A_2) \leq \sum_{k=n+1}^{2n+1} \mu(k, A_1 + A_2).$$

Suppose now that positive elements A_1 and A_2 are in $\mathcal{L}_{1,\infty}$. We have $A_1 + A_2 \in \mathcal{L}_{1,\infty}$ and, therefore,

$$\sum_{k=n+1}^{2n+1} \mu(k, A_1 + A_2) \leq \sum_{k=n+1}^{2n+1} \frac{\|A_1 + A_2\|_{1,\infty}}{k+1} \leq \|A\|_{1,\infty} \cdot \log(2).$$

Proofs. II

By the preceding slide, we have

$$\sum_{k=0}^{n} \mu(k, A_1 + A_2) = O(1) + \sum_{k=0}^{n} \mu(k, A_1) + \mu(k, A_2).$$

Hence,

$$\Big\{ \frac{1}{\log(n+2)} \sum_{k=0}^{n} \mu(k, A_1 + A_2) \Big\} \in c_0 +$$

$$+ \Big\{ \frac{1}{\log(n+2)} \sum_{k=0}^{n} \mu(k, A_1) \Big\} + \Big\{ \frac{1}{\log(n+2)} \sum_{k=0}^{n} \mu(k, A_2) \Big\}.$$

Since ω vanishes on c_0, the assertion follows.

Abstract description of Dixmier traces

Firstly, we define the Hardy-Littlewood pre-order $\prec\prec$. We say that $B \prec\prec A$ if

$$\sum_{k=0}^{n} \mu(k, B) \leq \sum_{k=0}^{n} \mu(k, A), \quad n \geq 0.$$

The assertion below is implicitly contained in KSS and SS.

Theorem

A positive normalised trace on $\mathcal{L}_{1,\infty}$ is Dixmier if and only if, on the positive cone of $\mathcal{L}_{1,\infty}$, it is monotone with respect to the Hardy-Littlewood pre-order.

Basic properties of traces

1. Every Dixmier trace is positive
2. Every positive trace is continuous with respect to the natural quasi-norm on $\mathcal{L}_{1,\infty}$
3. Every continuous trace is a linear combination of 4 positive traces.
4. There are positive traces which are not Dixmier traces
5. There exist discontinuous traces
6. There are $2^{2^{\mathbb{N}}}$ positive traces

Some of this is proved in subsequent sections.