# Lecture III

## Noncommutative geometry and “real” physics

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Noncommutative Geometry and Applications to Quantum Physics

# Action and Renormalization

The remarkable fact is that the fluctuations of the Dirac operator introduced the bosonic fields, gluons which are responsible for the strong (nuclear) force, the $\boxed{W}$ and $\boxed{Z}$ bosons responsible for the weak force, the photon and another field, which in view of its coupling to the fermion is responsible for the breaking of the symmetry and to give mass to the fermions.

This is the Higgs (Englert, Brout, Guralnick, Hagen, Kibble) boson

We should get numbers. In a form which can be confronted with experiment. And while we cannot aim at agreement with 12 significant digits, at least two or three...

## We have set the stage.

Ingredients:

- An algebra which describes a noncommutative (mild) generalization of spacetime, product of a continuous infinite dimensional part and a discrete finite dimensional noncommutative part
- A representation of the algebra on a Hilbert space containing the known fermions
- A generalized Dirac operator which has information on the curved background of the continuous Riemannian part as well as the information of the masses of the fermions for the discrete noncommutative part
- A chirality and a charge conjugation operator
- An action based on the spectrum of the operators, which we expand in series

Now we should just crack a machine

$$
\begin{gathered}
\mathcal{L}_{SM} = -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \\
\tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu (W^+_\mu \partial_\nu W^-_\mu - \\
W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \tfrac{1}{2} g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2} g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + \\
g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \tfrac{1}{2}\partial_\mu H \partial_\mu H - 2M^2 \alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
\beta_h \left(\tfrac{2M^2}{g^2} + \tfrac{2M}{g} H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \tfrac{2M^4}{g^2}\alpha_h - g\alpha_h M \left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) - \\
\tfrac{1}{8} g^2 \alpha_h \left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2\right) - gMW^+_\mu W^-_\mu H - \\
\tfrac{1}{2} g \tfrac{M}{c_w^2} Z^0_\mu Z^0_\mu H - \tfrac{1}{2} ig \left(W^+_\mu (\phi^0 \partial_\mu \phi^- - \phi^- \partial_\mu \phi^0) - W^-_\mu (\phi^0 \partial_\mu \phi^+ - \phi^+ \partial_\mu \phi^0)\right) + \\
\tfrac{1}{2} g \left(W^+_\mu (H\partial_\mu \phi^- - \phi^- \partial_\mu H) + W^-_\mu (H\partial_\mu \phi^+ - \phi^+ \partial_\mu H)\right) + \tfrac{1}{2} g \tfrac{1}{c_w} (Z^0_\mu (H\partial_\mu \phi^0 - \phi^0 \partial_\mu H) + \\
M\,(\tfrac{1}{c_w} Z^0_\mu \partial_\mu \phi^0 + W^+_\mu \partial_\mu \phi^- + W^-_\mu \partial_\mu \phi^+) - ig \tfrac{s_w^2}{c_w} M Z^0_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) + igs_w M A_\mu (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
ig \tfrac{1-2c_w^2}{2c_w} Z^0_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) + igs_w A_\mu (\phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+) - \tfrac{1}{4} g^2 W^+_\mu W^-_\mu \left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) - \\
\tfrac{1}{8} g^2 \tfrac{1}{c_w^2} Z^0_\mu Z^0_\mu \left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2 \phi^+\phi^-\right) - \tfrac{1}{2} g^2 \tfrac{s_w^2}{c_w} Z^0_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) - \\
\tfrac{1}{2} ig^2 \tfrac{s_w^2}{c_w} Z^0_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) + \tfrac{1}{2} g^2 s_w A_\mu \phi^0 (W^+_\mu \phi^- + W^-_\mu \phi^+) + \tfrac{1}{2} ig^2 s_w A_\mu H (W^+_\mu \phi^- - W^-_\mu \phi^+) - \\
g^2 \tfrac{s_w}{c_w} (2c_w^2 - 1) Z^0_\mu A_\mu \phi^+ \phi^- - g^2 s_w^2 A_\mu A_\mu \phi^+ \phi^- + \tfrac{1}{2} ig_s\, \lambda^a_{ij} (\bar{q}^\sigma_i \gamma^\mu q^\sigma_j) g^a_\mu - \bar{e}^\lambda (\gamma\partial + m^\lambda_e) e^\lambda - \bar{\nu}^\lambda (\gamma\partial + \\
m^\lambda_\nu) \nu^\lambda - \bar{u}^\lambda_j (\gamma\partial + m^\lambda_u) u^\lambda_j - \bar{d}^\lambda_j (\gamma\partial + m^\lambda_d) d^\lambda_j + igs_w A_\mu \left(-(\bar{e}^\lambda \gamma^\mu e^\lambda) + \tfrac{2}{3} (\bar{u}^\lambda_j \gamma^\mu u^\lambda_j) - \tfrac{1}{3} (\bar{d}^\lambda_j \gamma^\mu d^\lambda_j)\right) + \\
\tfrac{ig}{4c_w} Z^0_\mu \{(\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5) \nu^\lambda) + (\bar{e}^\lambda \gamma^\mu (4s_w^2 - 1 - \gamma^5) e^\lambda) + (\bar{d}^\lambda_j \gamma^\mu (\tfrac{4}{3} s_w^2 - 1 - \gamma^5) d^\lambda_j) + (\bar{u}^\lambda_j \gamma^\mu (1 - \tfrac{8}{3} s_w^2 + \\
\gamma^5) u^\lambda_j)\} + \tfrac{ig}{2\sqrt{2}} W^+_\mu \left((\bar{\nu}^\lambda \gamma^\mu (1+\gamma^5) {U^{lep}}_{\lambda\kappa} e^\kappa) + (\bar{u}^\lambda_j \gamma^\mu (1+\gamma^5) C_{\lambda\kappa} d^\kappa_j)\right) + \\
\tfrac{ig}{2\sqrt{2}} W^-_\mu \left((\bar{e}^\kappa {U^{lep}}^\dagger_{\kappa\lambda} \gamma^\mu (1+\gamma^5) \nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda} \gamma^\mu (1+\gamma^5) u^\lambda_j)\right) + \\
\tfrac{ig}{2M\sqrt{2}} \phi^+ \left(-m^\kappa_e (\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa} (1-\gamma^5) e^\kappa) + m^\lambda_\nu (\bar{\nu}^\lambda {U^{lep}}_{\lambda\kappa} (1+\gamma^5) e^\kappa)\right) + \\
\tfrac{ig}{2M\sqrt{2}} \phi^- \left(m^\lambda_e (\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa} (1+\gamma^5) \nu^\kappa) - m^\kappa_\nu (\bar{e}^\lambda {U^{lep}}^\dagger_{\lambda\kappa} (1-\gamma^5) \nu^\kappa)\right) - \tfrac{g}{2} \tfrac{m^\lambda_\nu}{M} H (\bar{\nu}^\lambda \nu^\lambda) - \tfrac{g}{2} \tfrac{m^\lambda_e}{M} H (\bar{e}^\lambda e^\lambda) + \\
\tfrac{ig}{2} \tfrac{m^\lambda_\nu}{M} \phi^0 (\bar{\nu}^\lambda \gamma^5 \nu^\lambda) - \tfrac{ig}{2} \tfrac{m^\lambda_e}{M} \phi^0 (\bar{e}^\lambda \gamma^5 e^\lambda) - \tfrac{1}{4}\, \bar{\nu}_\lambda\, M^R_{\lambda\kappa}\, (1-\gamma_5) \hat{\nu}_\kappa - \tfrac{1}{4}\, \overline{\bar{\nu}_\lambda\, M^R_{\lambda\kappa}\, (1-\gamma_5) \hat{\nu}_\kappa} + \\
\tfrac{ig}{2M\sqrt{2}} \phi^+ \left(-m^\kappa_d (\bar{u}^\lambda_j C_{\lambda\kappa} (1-\gamma^5) d^\kappa_j) + m^\lambda_u (\bar{u}^\lambda_j C_{\lambda\kappa} (1+\gamma^5) d^\kappa_j)\right) + \\
\tfrac{ig}{2M\sqrt{2}} \phi^- \left(m^\lambda_d (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1+\gamma^5) u^\kappa_j) - m^\kappa_u (\bar{d}^\lambda_j C^\dagger_{\lambda\kappa} (1-\gamma^5) u^\kappa_j)\right) - \tfrac{g}{2} \tfrac{m^\lambda_u}{M} H (\bar{u}^\lambda_j u^\lambda_j) - \tfrac{g}{2} \tfrac{m^\lambda_d}{M} H (\bar{d}^\lambda_j d^\lambda_j) + \\
\tfrac{ig}{2} \tfrac{m^\lambda_u}{M} \phi^0 (\bar{u}^\lambda_j \gamma^5 u^\lambda_j) - \tfrac{ig}{2} \tfrac{m^\lambda_d}{M} \phi^0 (\bar{d}^\lambda_j \gamma^5 d^\lambda_j)
\end{gathered}
$$

3

Here the notation is as in [46], as follows.

But we are not yet there...

The impressive lagrangian written above is still classical, one has to quantize it and implement on it the renormalization programme

Renormalization means that all constant quantities in the action are a functions of the energy: running coupling constants

The running is given by the solution of an ordinary differential equation, called the $\boxed{\beta}$ function, calculated perturbatively to first (occasionally second, rarely third) order in $\boxed{\hbar}$ by the $\boxed{n}$-point amplitude (loop expansion)

The three-point vertex insertion diagrams of class (ii) will similarly be denoted by (m/V), where m is the diagram of fig. 1 in which the insertion is made. The singular terms associated with these diagrams are given in table 2. The group theoretic factors introduced in these diagrams in addition to those in eqs. (3.3) – (3.10) are defined by

$$\overline{H}^{Y}_{abcd} \equiv \sum_{\text{perms}} \mathrm{Tr}\left(\underset{\sim}{Y}^{e}\underset{\sim}{Y}^{\dagger a}\underset{\sim}{Y}^{e}\underset{\sim}{Y}^{\dagger b}\underset{\sim}{Y}^{c}\underset{\sim}{Y}^{\dagger d}\right), \tag{3.13}$$

$$H^{\mathrm{S}}_{abcd} \equiv \sum_{k} C_2(k) H_{abcd}. \tag{3.14}$$

The four-point vertex insertion diagrams and the single pole diagrams, which will be analyzed together, are most conveniently classified into groups containing the same power of the gauge coupling constant.

The diagrams independent of the gauge coupling are shown in fig. 2, and the associated singular terms are given in table 3. The group theoretic factors associated with these diagrams are defined by

$$\Lambda^{3}_{abcd} \equiv \tfrac{1}{8} \sum_{\text{perms}} \lambda_{abef}\lambda_{efgh}\lambda_{ghcd}, \tag{3.15}$$

$$\overline{\Lambda}^{3}_{abcd} \equiv \tfrac{1}{4} \sum_{\text{perms}} \lambda_{abef}\lambda_{cegh}\lambda_{dfgh}, \tag{3.16}$$

$$H^{\lambda}_{abcd} \equiv \tfrac{1}{2} \sum_{\text{perms}} \lambda_{abef}\mathrm{Tr}\left(\underset{\sim}{Y}^{c}\underset{\sim}{Y}^{\dagger d}\underset{\sim}{Y}^{e}\underset{\sim}{Y}^{\dagger f}\right), \tag{3.17}$$

$$\overline{H}^{\lambda}_{abcd} \equiv \tfrac{1}{4} \sum_{\text{perms}} \lambda_{abef}\mathrm{Tr}\left(\underset{\sim}{Y}^{c}\underset{\sim}{Y}^{\dagger e}\underset{\sim}{Y}^{d}\underset{\sim}{Y}^{\dagger f}\right), \tag{3.18}$$

$$H^{3}_{abcd} \equiv \tfrac{1}{2} \sum_{\text{perms}} \mathrm{Tr}\left(\underset{\sim}{Y}^{a}\underset{\sim}{Y}^{\dagger b}\underset{\sim}{Y}^{e}\underset{\sim}{Y}^{\dagger c}\underset{\sim}{Y}^{d}\underset{\sim}{Y}^{\dagger e}\right). \tag{3.19}$$

TABLE 3
Singular parts of the scalar quartic vertex renormalization $Z_{abcd}$, as defined by eq. (3.1), for the diagrams shown in fig. 2 which are independent of the gauge coupling constant

| Diagram | $S_{abcd}$ | $A$ | $B$ |
|---|---|---|---|
| (2.1) | $\Lambda^{3}_{abcd}$ | $-\frac{1}{4}$ | 0 |
| (2.2) | $\overline{\Lambda}^{3}_{abcd}$ | $-\frac{1}{4}$ | $\frac{1}{4}$ |
| (2.3) | $\kappa H^{\lambda}_{abcd}$ | 2 | 0 |
| (2.4) | $\kappa \overline{H}^{\lambda}_{abcd}$ | 2 | $-2$ |
| (SP1) | $\kappa H^{3}_{abcd}$ | 0 | $-2$ |



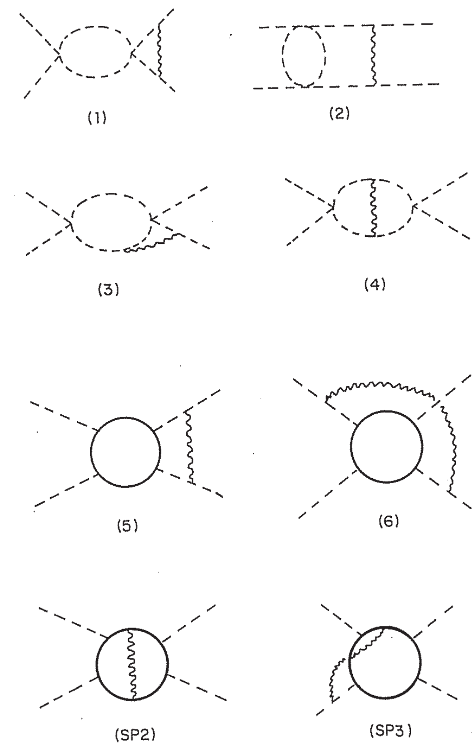



Fig. 3. Two-loop corrections to the proper scalar quartic vertex of order $g^2$.

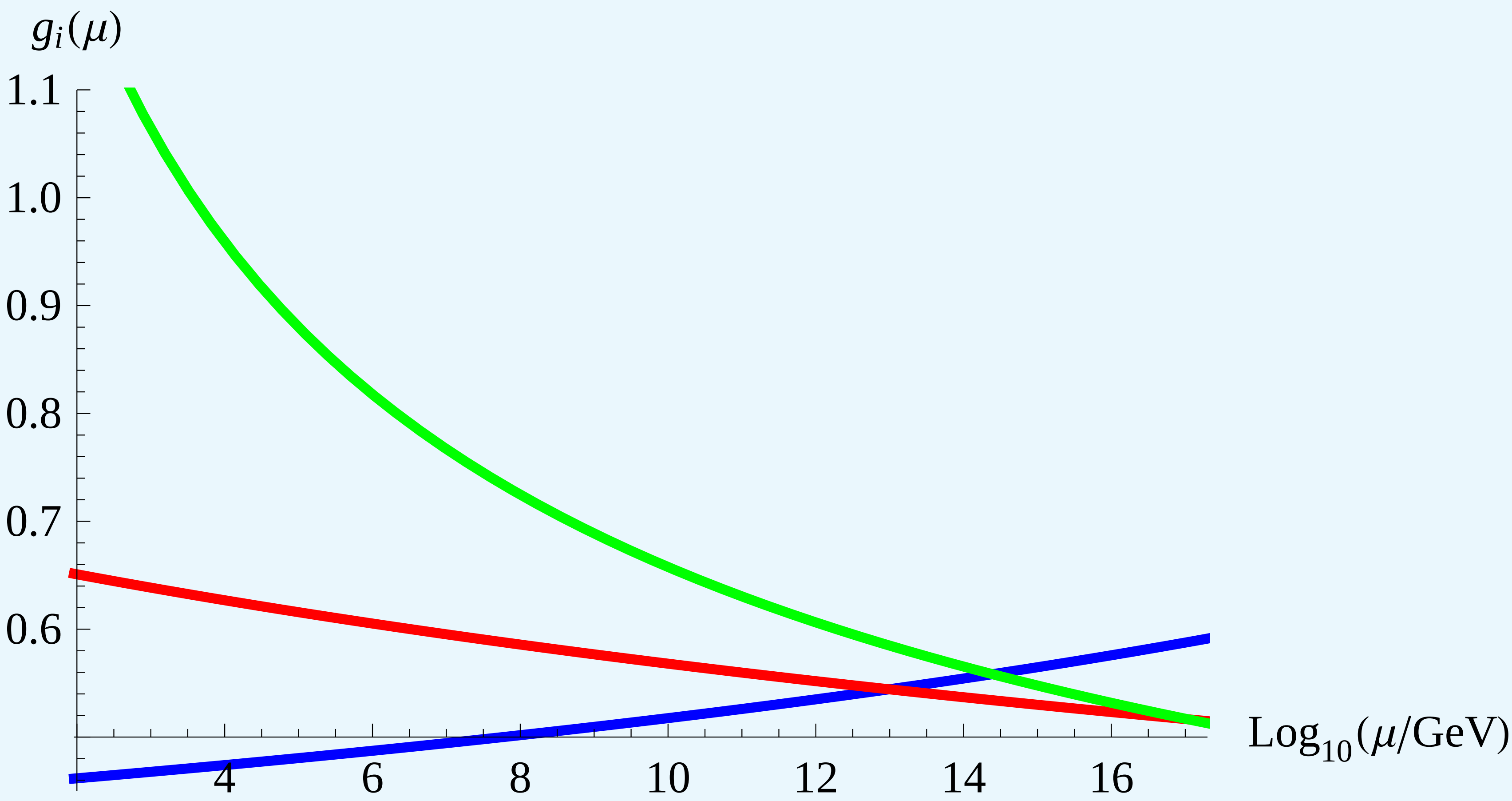
TABLE 4
Singular parts of the scalar quartic vertex renormalization $Z_{abcd}$, as defined by eq. (3.1), for the diagrams of order $g^2$ shown in fig. 3

| Diagram | $S_{abcd}$ | $A$ | $B$ |
|---|---|---|---|
| (3.1) | $\overline{\Lambda}^{2\mathrm{S}}_{abcd} + \Lambda^{2g}_{abcd} - \frac{3}{4}\Lambda^{2\mathrm{S}}_{abcd}$ | $-(1-\alpha)$ | 0 |
| (3.2) | $\overline{\Lambda}^{2\mathrm{S}}_{abcd} + \Lambda^{2g}_{abcd}$ | $\frac{1}{2}(1-\alpha)$ | $-\frac{1}{2}(1-\alpha)$ |
| (3.3) | $\overline{\Lambda}^{2\mathrm{S}}_{abcd} + \Lambda^{2g}_{abcd}$ | $1-\alpha$ | $1-\alpha$ |
| (3.4) | $\Lambda^{2g}_{abcd}$ | $-\frac{1}{2}(1-\alpha)$ | $\frac{1}{2}(2+\alpha)$ |
| (3.5) | $\kappa(H^{g}_{abcd} + H^{\mathrm{S}}_{abcd} - \frac{1}{2}H^{\mathrm{F}}_{abcd})$ | $-2(1-\alpha)$ | 0 |
| (3.6) | $\kappa(H^{g}_{abcd} + \frac{1}{2}H^{\mathrm{S}}_{abcd} - \frac{1}{2}H^{\mathrm{F}}_{abcd})$ | $2(1-\alpha)$ | $-2(1-\alpha)$ |
| (SP2) | $\kappa H^{g}_{abcd}$ | 0 | $-2(1-\alpha)$ |
| (SP3) | $\kappa(2H^{g}_{abcd} + H^{\mathrm{S}}_{abcd} - H^{\mathrm{F}}_{abcd})$ | 0 | $2(1-\alpha)$ |

The first think one has to decide is at which energy one write the big lagrangian. This will give a boundary condition

One boundary condition can be given by the fact that in the model obtained cranking the machine the strength of the fundamental interaction is equal

Experimentally is known that *if there are no other particles appearing at higher energy* the three coupling constant are almost equal in one point:



$g_i(\mu)$
1.1
1.0
0.9
0.8
0.7
0.6
$\text{Log}_{10}(\mu/\text{GeV})$
4
6
8
10
12
14
16

As I said the Dirac operator contains all data relative to the fermions, but no information on the Higgs mass (actually vev and quartic coupling coefficient) which can be calculated from the fermion mass parameters (Yukawa couplings). These in turn are dominated by the top quark coupling.

Hence we have a "prediction" for the Higgs mass.

The prediction is $\boxed{175.1 + 5.8 - 7.2\text{GeV}}$.

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**This number is wrong.**

The actual experimental value is $\boxed{125.09 \pm 32\text{GeV}}$.

Let me make a little sociological comment. In these lectures, especially this last one, I have been talking with very little mathematical rigour. This may disconcert, or even horrify mathematicians. Yet, in physics we have a very stringent rigour: experimental verification. We let ourselves do anything we want with a theory, and in the end we judge it by its predictions.

Now it depends how you consider this theory. if you take it as a mature fully formed theory then the result is wrong. That's it, throw away the theory.

If you take it (as I do) as a tool to investigate the standard model starting from first principles, then I think it is remarkable that a theory based on pure mathematical result gets reasonable numbers

Alternatively take the measurement of the Higgs as a reason to understand in which direction one has to improve on the theory. And hasten to add that it possible to reconcile it with experiment.

Let me sketch the reasons for the wrong prediction. A complete explanation would bring us afar and be difficult to follow for mathematicians

The prediction depends on the boundary condition, and we were forced to use one for which the coupling constants of the three interactions were equal at a scale. This is true only approximately, but the picture I showed is based on a running which starts from low energy (where we have data) and extrapolates to high energy.

But if one changes the field content, then the runnings of *all* quantities change

Enter right handed neutrinos. The most recent addition to the particle zoo. Their mass should be very high to give their left handed partners an incredibly small mass (via the so called see-saw mechanism). It turns out that if in place of the coupling of these neutrons one puts (by hand) a field, then the running changes in such a way that the model becomes compatible with experimental data, and in passing also solves a dangerous instability of the potential

But unfortunately the addition of this extra field loosens the predictive power, and does not really fit in the scheme as I described so far. The cracking of the machine does not really give it. But we can use it to understand something more of the noncommutative geometry of the standard model.

Since the conditions for a spectral triple to describe a manifold have been cast algebraically, we can see which noncommutative finite dimensional $C^*$ algebras satisfy the conditions. And I remind you that a finite dimensional $C^*$ is necessarily a sum of matrices over the reals, complex or quaternions

This is a straightforward exercise you. But you need use all of the five elements of the triple. The result is that the finite part of the spectral triple must have a well defined form:

$$\mathbb{M}(\mathbb{H})_a \oplus \mathbb{M}(\mathbb{C})_{2a}$$

for $a$ integer.

The direct sum of matrices of quaternions (which in turn can be represented as $2 \times 2$ matrices) and matrices of complex number of the same size.

We need the algebra to be represented on an Hilbert space of dimension $\boxed{n = 2(2a)^2}$ (up to generation replicas)

The gauge group of this algebra is made of the unitary operators, and the symmetry will be "broken", thus reducing the gauge group.

Hence there is not much freedom in the game, since you need to see if there is way to obtain the algebra of the standard model

This algebra acts on a finite Hilbert space of dimension $\boxed{2(2a)^2}$.
For a non trivial grading it must be $\boxed{a \geq 2}$

$$\boxed{\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})}$$

Hence an Hilbert space of dimension $\boxed{2(2 \cdot 2)^2 = 32}$, the dimension of $\boxed{\mathcal{H}_F}$ for one generation.

The grading condition $\boxed{[a, \Gamma] = 0}$ reduces the algebra to the left-right algebra

$$\boxed{\mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C})}$$

The order one condition reduces further the algebra to $\mathcal{A}_{sm}$, i.e. the algebra whose unimodular group is U(1)×SU(2)×U(3)

The group $\boxed{SU(4) \times SU(2) \times SU(2)}$ has been introduced long ago (Pati-Salam). Is one of the first example of a Grand Unified Theory. There is a sort of fourth colour (lepton number) and is left-right symmetric.

This unified theory should break to the standard model. A field called $\boxed{\sigma}$ (analog to the Higgs) is necessary. This field appears in $\boxed{D}$ in the position corresponding to a particular form of the neutrino mass (Majorana). It turns out that precisely in that spot (and not many others) it is possible to put a nonzero value!

But cranking of the machine does not produce a contribution to the one form, the extra term commutes with the algebra. Hence must included it by hand. Which is unpleasant

Doing again the running of the physical quantitates with this field does change the Higgs mass, making it compatible with the experimental value

Physics is therefore telling us that into his framework right handed neutrinos, and Majorana masses are crucial

Can we avoid adding this field by hand? There are three possible solutions

- Enlarge the Hilbert space introducing new fermions and new interactions. Stephan

- Consider a Grand Symmetry based on $\boxed{\mathbb{M}(\mathbb{H})_4 \oplus \mathbb{M}(\mathbb{C})_8}$ Devastato Lizzi Martinetti

- Violate one of the conditions (order one) Chamseddine, Connes Van Suijlekom

The latter solutions allow the introduction of a new field $\boxed{\sigma}$ which not only fixes the mass of the Higges making it compatible with 126 GeV, but also solves the possible instability of the theory.

A. Devastato, P. Martinetti and I proposed as solution: a grand symmetry.

In NCG the usual grand unified groups, such as $\boxed{SU(5)}$ or $\boxed{SO(10)}$ do not work. There are very few representations of algebra as opposed to groups. Finite dimensional algebras only have one nontrivial IRR

Fortunately in the standard model there are only weak doublets and colour triplets, so it works

Recall that a finite "manifold" is an algebra: $\boxed{M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C})}$ acting on a $\boxed{2(2a)^2}$ dimensional Hilbert space. So far we had $\boxed{a = 2,\ 2(2a)^2 = 32 \times 3 = 96}$

The numerology comes out correct

For $M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$ one requires a $2(2 \cdot 4)^2 = 128$ dimensional space. (384 taking generations into account)

This is exactly the dimension of the Hilbert space if we take the fermion doubling into account. This overcounting had been perceived as a nuisance if not a problem. One had to project states out, and the unphysical redundancy was unexplained

It is necessary to look at Hilbert space with different eyes

$$\mathcal{H} = sp(L^2(M)) \otimes \mathcal{H}_F = L^2(M) \otimes \mathsf{H}_F$$

where now the dimensions of $\mathsf{H}_F$ is 384

It is still possible to represent the gran algebra $\boxed{M_4(\mathbb{H}) \oplus M_8(\mathbb{C})}$ satisfying all of the manifold conditions. This is highly nontrivial if one keeps the same Hilbert space.

But this time the algebra does not act diagonally on the spinor indices. it mixes them. In addition one has to consider a particular form of spectral triple: twisted triples, for which the commutators present in the various conditions are twisted by an isomorphism of the algebra.

If I have no time I will not write explicitly the details of the representation (on particle and anti particles) because they are rather involved. The key point is that in the process spacetime indices, related to the Euclidean symmetries, mix with internal, gauge indices.

We envisage this Grand Symmetry to belong to a pre geometric phase. At this stage all elements of $D_F$ may be negligible, and the spinor part of the direct operator $\partial\!\!\!/$ will cause the "breaking" to a phase in which the symmetries of the phase space emerge

In particular, the order one condition for $\partial\!\!\!/$ causes the reduction of the algebra to $\mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})$

And there is an added bonus:

This grand algebra, and a corresponding $\boxed{D}$ operator, have "more room" to operate. Although the Hilbert space is the same, the fact that we abandoned the factorization of the internal indices, gives us more entries to accommodate the Majorana masses

Hence we can put a Majorana mass for the neutrino and at the same time satisfy the order one condition. Then the one form corresponding to this $\boxed{D_\nu}$ will give us the by now famous field $\boxed{\sigma}$, which can only appear before the transition to the geometric spacetime

The natural scale for this mass is to be above a transition which gives the geometric structure. Therefore it is natural that it may be at a high scale. How high we can discuss

The grand symmetry is no ordinary gauge symmetry, there is never a $\boxed{SU(8)}$ in the game for example

It represents a phase in which the internal noncommutative geometry contains also the spin structure, even the Lorentz (Euclidean) structure of space time in a mixed way

The differentiation between the spin structure of spacetime, and the internal gauge theory comes as a breaking of the symmetry, triggered by $\boxed{\sigma}$, which now appears naturally has having to do with the geometry of spacetime.

The fermion doubling was not a problem after all. . .

# Euclid vs. Lorentz

The use of Euclidean actions if field theory is also common. What is usually said if that "in the end you Wick rotate to Lorentz signature".

WIck rotation is a procedure to change the signature of field theory. It consists 8loosely speaking) in "rotating" the time derivative in the complex plane $\boxed{t \to it}$. This changes the signature of space time from a Lorentzian metric to a Euclidean one

This renders some integrals, whoch would be oscillatory in the functional integration, convergent since $\boxed{e^{it} \to e^{-t}}$. In some cases other regularizations work as well, and in principle they are just equivalent procedure which can work always, even if the technical difficulties can be very different

Then one Wick rotates back, i.e., undoes an operation. But in the spectral approach we cannot start unless we have an Euclidean theory. So we are not going back, we are going in unchartered territory

Usually a Wick rotation is indicated as the transformation $\boxed{t \to \mathrm{i}t}$, even if a more correct procedure would be to rotate the vierbein. Namely for each $F$, which depends on vierbeins

$$\boxed{\text{Wick:} \quad F\left[e^0_\mu, e^j_\mu\right] \longrightarrow \quad F\left[\mathrm{i}e^0_\mu, e^j_\mu\right], \quad j = 1, 2, 3.]}$$

The inverse (which is what usually people call Wick rotation) is

$$\boxed{\text{Wick}^* : \quad F\left[e^0_\mu, e^j_\mu\right] \longrightarrow \quad F\left[-\mathrm{i}e^0_\mu, e^j_\mu\right] \quad j = 1, 2, 3.}$$

For the bosonic part of the spectral action things go relatively without problems, the prescription is clear and the action is rotated into a new one which makes the partition function convergent

$$\boxed{\text{Wick:} \quad S^{\mathrm{E}}_{\mathrm{bos}}\left[\text{fields}, \mathrm{g}^{\mathrm{E}}_{\mu\nu}\right] \longrightarrow S^{\mathrm{E}}_{\mathrm{bos}}\left[\text{fields}, -\mathrm{g}^{\mathrm{M}}_{\mu\nu}\right] \equiv -\mathrm{i}S^{\mathrm{M}}_{\mathrm{bos}}\left[\text{fields}, \mathrm{g}^{\mathrm{M}}_{\mu\nu}\right]}$$

## The fermionic sector requires some extra considerations

The group $\boxed{\mathsf{Spin}(1,3)}$ is quite different from $\boxed{\mathsf{Spin}(4)}$, $\boxed{\gamma}$ matrices, generators, charge conjugation, change. Also the fermionic action changes, since the quadratic forms have to be invariant under the proper group transformations

$$\boxed{\bar{\psi}\,\gamma^A_{\mathsf{M}}\,e^\mu_A\left(\left[\nabla^{\mathsf{LC}}_\mu\right]^{\mathsf{M}} + \mathrm{i}A_\mu\right)\psi, \quad \bar{\psi}\psi}$$

$\boxed{\bar{\psi} \equiv \psi^\dagger\gamma^0}$ and $\boxed{\nabla^{\mathsf{LC}}_\mu}$ the covariant derivative on the spinor bundle with the Levi-Civita spin-connection, which is different for Lorentzian and Euclidean

The corresponding terms with the required Spin(4) invariance are:

$$\boxed{\psi^\dagger\,\gamma^A_{\mathsf{E}}\,e^\mu_A\left[\nabla^{\mathsf{LC}}_\mu\right]^{\mathsf{E}}\psi, \quad \psi^\dagger\psi}$$

The charge conjugations are:

$$C_{\mathsf{M}}\psi = -\mathrm{i}\gamma^2_{\mathsf{M}}\psi^* \quad ; \; C_{\mathsf{E}}\psi = \mathrm{i}\gamma^0_{\mathsf{E}}\gamma^2_{\mathsf{E}} = \widehat{C}_{\mathsf{E}}\psi^*$$

The Majorana mass term is the same in both cases:

$$\underbrace{(C_{\mathsf{E}}\psi)^\dagger\,\psi}_{Spin(4)\ \text{inv}} = (-i\gamma^0_{\mathsf{E}}\gamma^2_{\mathsf{E}}\psi^*)^\dagger\psi = \overline{(\gamma^2_{\mathsf{M}}\psi^*)}\psi = -\underbrace{i\,\overline{(C_{\mathsf{M}}\psi)}\psi}_{Spin(1,3)\ \text{inv}}$$

Also the spacetime grading is the same in the two cases

$$\gamma^5 = \gamma^0_{\mathsf{E}}\gamma^1_{\mathsf{E}}\gamma^2_{\mathsf{E}}\gamma^3_{\mathsf{E}} = \mathrm{i}\gamma^0_{\mathsf{M}}\gamma^1_{\mathsf{M}}\gamma^2_{\mathsf{M}}\gamma^3_{\mathsf{M}}$$

so that the definition of left and right spinor is the same

The difference between $\psi^\dagger$ which appers in the Euclidean, and the Lorentzian $\bar{\psi}$ is the presence of a $\gamma^0$ which must be inserted in the Lorentzian case

In NCG the fermionic spectral action is

$$S_F = \frac{1}{2}\langle J\psi, D_A\psi\rangle$$

Thanks to the extra degrees of freedom, the insertion of $\gamma^0$ by hand is not needed for this action, which therefore deals with slightly different structures.

The fermionic action is build in any case contracting the a conjugate spinor with an operator acting on a spinor. Let us look at the charge conjugation

The spacetime part of the Hilbert space splits into eigenspaces of chirality, each of which has two components, for particles and antiparticles

$$\boxed{\mathrm{Sp}(M) = H_{\mathcal{L}} \oplus H_{\mathcal{R}}}$$

with our conventions a the antiparticle of a left particle is right, and viceversa

At the same time the internal space has a similar decomposition given by the internal grading $\boxed{\gamma}$

$$\boxed{H_F = H_L \oplus H_R \oplus H_L^c \oplus H_R^c}$$

One problem with the quadruplication is the presence of "mirrors", states which have different chiralities. They have to be projected out, defining $\boxed{\mathcal{H}_+}$

$$\boxed{H_+ = (H_L)_{\mathcal{L}} \oplus (H_R)_{\mathcal{R}} \oplus (H_L^c)_{\mathcal{R}} \oplus (H_R^c)_{\mathcal{L}} = P_+ H, \quad P_+ \equiv \frac{\mathbb{I} + \Gamma}{2}}$$

This takes care of half of the extra degrees of freedom. The fermionic action is then defined as

$$S_F = \frac{1}{2}\langle J\psi, D_A\psi\rangle \quad \psi \in H_+$$

with $J = C_{\mathsf{E}} \otimes J_F$ and $J_F = \begin{pmatrix} 0 & 0 & \mathbb{I} & 0 \\ 0 & 0 & 0 & \mathbb{I} \\ \mathbb{I} & 0 & 0 & 0 \\ 0 & \mathbb{I} & 0 & 0 \end{pmatrix} \circ cc.$

The action reproduces correctly the Pfaffian i.e. functional integral over fermions, but this procedure only takes care of half of the extra degrees of freedom. In processes like scattering, after quatization, it is important to have the correct Hilbert space of incoming and outgoing particles.

In the bosonic spectral action the operator $D$ is present, not $DP_+$, which is not Hermitian and not a square root of the Laplacian.

The extra degrees of freedom are taken care by the Wick rotation. It is in fact necessary to first perform the Wick rotation in order to eliminate the charge conjugation doubling

A naive attempt to remove it from the action with the $\boxed{J}$ would break the Euclidean Spin(4) symmetry.

Only the combination of Wick rotation (and identification of states described below) and the projection renders the action viable for physical applications, and free of the fermion doubling

We can skip the details if there is no time

What have we learned? I think the most intriguing element is that the Euclidean fermionic action, which uses in a crucial way the real structure of the spectral triple, and needs the fermionic quadruplication, is naturally rotated in the Lorentzian, with the elimination of the extra degrees of freedom.

There are various studies which connect spectral triples and Lorentz signatures Verch, Paschke, Sitarz, Eckstein, Franco, Besnard, Bizi, Van den Dungen, . . . . The considerations I wxposed suggest that a possible way to obtain Lorentzian spectral triple is a rigorous treatment of Wick rotations.

# Conclusions

Noncommutative geometry starts with a view of geometry based on spectral properties, and is geared towards a profound generalization historically opened by the necessity to describe the quantum world

But then noncommutative grows to become mora a philosophy for which what is fundamental are not anymore the points, but rather the algebraic structures that we can build over them.

I tried in the last two lectures to give you the flavour of an application to the physics of fundamental interactions. What we are doing is to understand the noncommutative geometry of the standard model. This view is not the "party line" of particle physicists, but nevertheless not only gives a more general framework, which may lead to a more profound understanding, but also makes it conceivable that it may an actual contribution to phenomenology, and confront itself with experiments in a positive way.

# References

Apart from the ones of the previous lecture

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# Details of the representation of the Grand Algebra

There are several finite dimensional algebras in this game, and I want to look at their representations

Ultimately we want to ge to the the standard model algebra

$$\mathcal{A}_{sm} = \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{M}_3(\mathbb{C}),$$

$\mathbb{H}$ are the quaternions, which we represent as $2 \times 2$ matrices

It is possible to have this emerge from the most general algebra which satisfies the condition of being a noncommutative manifold

$$\mathcal{A}_{\mathcal{F}} = \mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C}) \qquad a \in \mathbb{N}.$$

This algebra acts on a finite Hilbert space of dimension $\boxed{2(2a)^2}$.
For a non trivial grading it must be $\boxed{a \leq 2}$

$$\boxed{\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})}$$

Hence an Hilbert space of dimension $\boxed{2(2 \cdot 2)^2 = 32}$, the dimension of $\boxed{\mathcal{H}_F}$ for one generation.

The grading condition $\boxed{[a, \Gamma] = 0}$ reduces the algebra to the left-right algebra

$$\boxed{\mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C})}$$

The order one condition reduces further the algebra to $\mathcal{A}_{sm}$, i.e. the algebra whose unimodular group is U(1)×SU(2)×U(3)

Let us look in detail to a vector in the Hilbert space:

$$\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x) \in \mathcal{H} = L^2(\mathcal{M}) \otimes \mathsf{H}_F = sp(L^2(\mathcal{M})) \otimes \mathcal{H}_F$$

Note the difference between $\mathcal{H}_F$, which is 96 dimensional, and $\mathsf{H}_F$ which is 384 dimensional. The meaning of the indices is as follows:

$$\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x)$$

$$\begin{aligned} s &= r, l \\ \dot{s} &= \dot{0}, \dot{1} \end{aligned}$$

are the spinor indices. They are not internal indices in the sense that the algebra $\mathcal{A}_F$ acts diagonally on it. They take two values each, and together they make the four indices on an ordinary Dirac spinor.

$$\Psi^{\mathsf{c}\mathrm{I}m}_{s\dot{s}\alpha}(x)$$

$\mathrm{I} = 0, \ldots 3$ indicates a "lepto-colour" index. The zeroth "colour" actually identifies leptons while $\mathrm{I} = 1, 2, 3$ are the usual three colours of QCD.

$$\Psi^{\mathsf{c}\mathrm{I}m}_{s\dot{s}\alpha}(x)$$

$\alpha = 1 \ldots 4$ is the flavour index. It runs over the set $u_R, d_R, u_L, d_L$ when $\mathrm{I} = 1, 2, 3$, and $\nu_R, e_R, \nu_L, e_L$ when $\mathrm{I} = 0$. It repeats in the obvious way for the other generations.

$$\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x)$$

$\mathsf{C} = 0, 1$ indicates whether we are considering "particles" ($\mathsf{C} = 0$) or "antiparticles" ($\mathsf{C} = 1$).

$$\Psi^{\mathsf{C}\mathrm{I}m}_{s\dot{s}\alpha}(x)$$

$m = 1, 2, 3$ is the generation index. The representation of the algebra of the standard model is diagonal in these indices, the Dirac operator is not, due to Cabibbo-Kobayashi-Maskawa mixing parameters. For this seminar plays no role, and will ignored.

We can now give explicitly the algebra representations in term of these indices.

We start from $\mathcal{A}_F = \mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})$, a generic element will depend on $4 \times 4$ complex matrix $m$, and a $2 \times 2$ matrix of quaternions $q$, which we may also see as a $4 \times 4$ with some conditions

The representation in its fullness is

$$A^{t\dot{t}\,\mathsf{CI}\beta}_{s\dot{s}\mathsf{DJ}\alpha} = \delta^t_s \delta^{\dot{t}}_{\dot{s}} \left( \delta^{\mathsf{C}}_{0} \delta^{\mathsf{I}}_{\mathsf{J}} Q^{\beta}_{\alpha} + \delta^{\mathsf{C}}_{1} M^{\mathsf{I}}_{\mathsf{J}} \delta^{\beta}_{\alpha} \right)$$

Note the two $\delta$'s at the beginning which show that the algebra acts trivially on the spacetime indices, and the fact that the two matrices act on different indices. This ensures the order zero condition, namely exchanging particles with antiparticles, the job done by $J$, the two representations commute.

The representations of the other algebra are similar, in the case of the standard model there is a differentiation with the leptocolour indices.

The order one condition and a $\boxed{\nu}$ Majorana mass cause the reduction to $\boxed{C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}}$, represented as

$$\boxed{a = \{m, q, c\} \text{ with } m \in C^\infty(\mathcal{M}) \otimes \mathbb{M}_3(\mathbb{C}),\ q \in C^\infty(\mathcal{M}) \otimes \mathbb{H},\ c \in C^\infty(\mathcal{M}) \otimes \mathbb{C}}$$

is

$$\boxed{a^{t\dot{t}\mathsf{C}\mathrm{I}\beta}_{s\dot{s}\mathsf{D}\mathsf{J}\alpha} = \delta^t_s \delta^{\dot{t}}_{\dot{s}} \left(\delta^{\mathsf{C}}_0 \delta^{\mathrm{I}}_{\mathsf{J}} \left(\mathsf{q}^\beta_\alpha + \mathsf{c}^\beta_\alpha\right) + \delta^{\mathsf{C}}_1 \left(\mathsf{m}^{\mathrm{I}}_{\mathsf{J}} + \tilde{\mathsf{c}}^{\mathrm{I}}_{\mathsf{J}}\right) \delta^\beta_\alpha\right)}$$

where we use the following $\boxed{4 \times 4}$ complex matrices:

$$\mathsf{q} = \begin{pmatrix} 0_2 & \\ & q \end{pmatrix}_{\alpha\beta}, \qquad \mathsf{c} = \begin{pmatrix} c & & \\ & \bar{c} & \\ & & 0_2 \end{pmatrix}_{\alpha\beta}, \qquad \tilde{\mathsf{c}} = \begin{pmatrix} c & \\ & 0_3 \end{pmatrix}_{\mathrm{IJ}}, \qquad \mathsf{m} = \begin{pmatrix} 0 & \\ & m \end{pmatrix}_{\mathrm{IJ}}$$

The breaking $\mathcal{A}_F$ to $\mathcal{A}_{sm}$ goes with the chirality and first order conditions

I can similarly write down the Dirac operator

$$D = \partial\!\!\!/ \otimes \mathbb{I}_{96} + \gamma^5 \otimes D_F$$

$$D_F = \begin{pmatrix} 0_{8N} & \mathcal{M} & \mathcal{M}_R & 0_{8N} \\ \mathcal{M}^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ \mathcal{M}_R^\dagger & 0_{8N} & 0_{8N} & \bar{\mathcal{M}} \\ 0_{8N} & 0_{8N} & \mathcal{M}^T & 0_{8N} \end{pmatrix}.$$

$\mathcal{M}$ contains the Dirac-Yukawa couplings. It links left with right particles. $\mathcal{M}_R = \mathcal{M}_R^T$ contains Majorana masses and links righr particles with right antiparticles. $\mathcal{M} = \begin{pmatrix} M_u & 0_{4N} \\ 0_{4N} & M_d \end{pmatrix} \qquad \mathcal{M}_R = \begin{pmatrix} M_R & 0_{4N} \\ 0_{4N} & 0_{4N} \end{pmatrix}$ where $M_u$ contaimins the masses of the up, charm and top quarks and the neutrinos (Dirac mass), $M_R$ contains the Majorana neutrinos masses and $M_d$ the remaining quarks and electrons, muon and tau masses, including mixings

I think by now you know the rules. With the algebra and $D$ one builds the one-form, which are the fluctuations of the Dirac operator.The bosonic fields are coming from these one-form $\sum_i a_i[D, b_i]$

But here we run into a problem: the elements of $\mathcal{M}_R$ are the ones which should give rise to the field $\sigma$ as intermediate boson, on a par with the Higgs, and relate to the breaking of the left-right symmetry.

Except that this term either commutes with $D$ or violates the first order condition!

One alternative would is to have a combination of algebra and Dirac operator violating the first order condition

Or we may look for a bigger algebra...

Consider the case of $\mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C})$ for the case $a = 4$

In this case we need a $2 \cdot (2 \cdot 4)^2 = 128$ dimensional space, which for 3 generations gives a $384$ dimensional Hilbert space.

I need a representation of the algebra $\mathbb{M}_4(\mathbb{H}) \oplus \mathbb{M}_8(\mathbb{C})$ acting on the spinors I gave earlier, and which can satisfy the stringent order zero conditions

Consider $Q \in \mathbb{M}_4(\mathbb{H})$ and $M \in \mathbb{M}_8(\mathbb{C})$ with indices

$$Q^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} Q^{\dot{0}\beta}_{\dot{0}\alpha} & Q^{\dot{1}\beta}_{\dot{0}\alpha} \\ Q^{\dot{0}\beta}_{\dot{1}\alpha} & Q^{\dot{1}\beta}_{\dot{1}\alpha} \end{pmatrix}_{\dot{s}\dot{t}}, \qquad M^{t\mathrm{I}}_{s\mathrm{J}} = \begin{pmatrix} M^{r\mathrm{I}}_{r\mathrm{J}} & M^{l\mathrm{I}}_{r\mathrm{J}} \\ M^{r\mathrm{I}}_{l\mathrm{J}} & M^{l\mathrm{I}}_{l\mathrm{J}} \end{pmatrix}_{st}$$

where, for any $\dot{s}, \dot{t} \in \{\dot{0}, \dot{1}\}$ and $s, t \in \{l, r\}$, the matrices

$Q^{\dot{t}\beta}_{\dot{s}\alpha} \in \mathbb{M}_2(\mathbb{H}), \quad M^{t\mathrm{I}}_{s\mathrm{J}} \in \mathbb{M}_4(\mathbb{C})$ have the index structure defined above

The representation of the element $\boxed{A = (Q, M) \in \mathcal{A}_G}$ is:

$$\boxed{\mathsf{A}^{t\dot{t}\mathsf{C}\;\mathrm{I}\beta}_{s\dot{s}\mathsf{D}\mathsf{J}\alpha} = \left(\delta^{\mathsf{C}}_{0}\delta^{t}_{s}\delta^{\mathrm{I}}_{\mathsf{J}}\,Q^{\dot{t}\beta}_{\dot{s}\alpha} + \delta^{\mathsf{C}}_{1}M^{t\mathrm{I}}_{s\mathsf{J}}\delta^{\dot{t}}_{\dot{s}}\delta^{\beta}_{\alpha}\right)}$$

compare with the previous case

$$\boxed{A^{t\dot{t}\,\mathsf{C}\mathrm{I}\beta}_{s\dot{s}\mathsf{D}\mathsf{J}\alpha} = \delta^{t}_{s}\delta^{\dot{t}}_{\dot{s}}\left(\delta^{\mathsf{C}}_{0}\delta^{\mathrm{I}}_{\mathsf{J}}Q^{\beta}_{\alpha} + \delta^{\mathsf{C}}_{1}M^{\mathrm{I}}_{\mathsf{J}}\delta^{\beta}_{\alpha}\right)}$$

The spinor indices and the internal gauge indices are mixed. We are in a phase in which the Euclidean structure of space time has not yet emerged.

The fermions are not yet fermions

back

# Details of the Wick rotation

First we rotate the action as in the bosonic case:

$$\text{Wick rotation:} \quad -S_F^{\mathsf{E}}\left[\text{spinors}, e_\mu^A\right] \longrightarrow \mathrm{i} S_F^{\mathsf{M}\,\text{doubled}}\left[\text{spinors}, e_\mu^A\right]$$

We now have a Lorentz invariant fermionic action invariant under $Spin(1,3)$ but still exhibiting a doubling. The spinors are in $H_+$, which is not anymore a Hilbert space with respect to the $Spin(1,3)$ invariant inner product

The remaining doubling consists in presence of spinors from all four subspaces of $H_+$: $\left(H_L^c\right)_{\mathcal{R}}, \left(H_R^c\right)_{\mathcal{L}}, (H_L)_{\mathcal{L}}, (H_R)_{\mathcal{R}}$

The physical Lagrangian depends on spinors just from the last two

After the Wick rotation we should perform the following identification

$$\begin{cases} \left(\psi_L^c\right)_{\mathcal{R}} \in \underbrace{(H_L^c)_{\mathcal{R}}}_{\in H_+} \quad \text{identified with} \quad C_{\mathsf{M}}\,(\psi_L)_{\mathcal{L}}\,, \quad (\psi_L)_{\mathcal{L}} \in \underbrace{(H_L)_{\mathcal{L}}}_{\in H_+} \\ \left(\psi_R^c\right)_{\mathcal{L}} \in \underbrace{(H_R^c)_{\mathcal{L}}}_{\in H_+} \quad \text{identified with} \quad C_{\mathsf{M}}\,(\psi_R)_{\mathcal{R}}\,, \quad (\psi_R)_{\mathcal{R}} \in \underbrace{(H_R)_{\mathcal{R}}}_{\in H_+} \end{cases} .$$

This step leads to the same formula of Barrett, who started directly Lorentzian.

We can then apply the procedure to the spectral action:

First we restore Lorentz signature in the action

$$
\begin{aligned}
-S_F^{\mathsf{E}} \;\to\; & -\int d^4x\sqrt{-g^{\mathsf{M}}}\begin{bmatrix} C_{\mathsf{E}}\left(\psi_L^c\right)_{\mathcal{R}} \\ C_{\mathsf{E}}\left(\psi_R^c\right)_{\mathcal{L}} \end{bmatrix}^{\dagger}\begin{bmatrix} \mathrm{i}\nabla\!\!\!/^{\mathsf{M}} & \mathrm{i}M_D \\ \mathrm{i}M_D^{\dagger} & \mathrm{i}\nabla\!\!\!/^{\mathsf{M}} \end{bmatrix}\begin{bmatrix} (\psi_L)_{\mathcal{L}} \\ (\psi_R)_{\mathcal{R}} \end{bmatrix} \\
& -\frac{\mathrm{i}}{2}\int d^4x\sqrt{-g^{\mathsf{M}}}\left\{\left[C_{\mathsf{E}}\,(\psi_R)_{\mathcal{R}}\right]^{\dagger} M_M\,(\psi_R)_{\mathcal{R}} + \left[C_{\mathsf{E}}\left(\psi_R^c\right)_{\mathcal{L}}\right]^{\dagger} M_M^{\dagger}\left(\psi_R^c\right)_{\mathcal{L}}\right\}
\end{aligned}
$$

This action is Lorentz invariant under. No modification of the inner product, like the insertion of $\boxed{\gamma^0}$, is needed.

Since $\boxed{C_{\mathsf{E}} = \mathrm{i}\gamma^0 C_{\mathsf{M}}}$ we have the manifestly Lorentz invariant action:

$$
\boxed{
\begin{aligned}
S_F^{\mathsf{M}} \;=\; & \int d^4x\sqrt{-g^{\mathsf{M}}}\Big\{\overline{(\psi_{\mathcal{L}})}\,\mathrm{i}\nabla\!\!\!/^{\mathsf{M}}\psi_{\mathcal{L}} + \overline{(\psi_{\mathcal{R}})}\,\mathrm{i}\nabla\!\!\!/^{\mathsf{M}}\psi_{\mathcal{R}} \\
& - \Big[\overline{(\psi_{\mathcal{L}})}\,H\,\psi_{\mathcal{R}} + \tfrac{1}{2}\overline{[C_{\mathsf{M}}\,(\psi_{\mathcal{R}})]}\,\omega\,\psi_{\mathcal{R}} + \text{c.c.}\Big]\Big\}
\end{aligned}
}
$$

We still have extra degrees of freedom since each quantity which carries the index "$c$" is independent from the one which does not.

It is remarkable that the path integral is not sensitive to the charge conjugation doubling, in particular the Pfaffian is reproduced correctly since

$$\int [d\bar{\psi}][d\psi] e^{\mathrm{i} \int d^4x \, \bar{\psi} \, \mathrm{i}\not{\partial}^{\mathrm{M}} \psi} = \int [d\bar{\xi}][d\psi] e^{\mathrm{i} \int d^4x \, \bar{\xi} \, \mathrm{i}\not{\partial}^{\mathrm{M}} \psi}.$$

The correct identification of the Hilbert space is necessary. The Lorentzian theory has to be *quantized*, and the quantum Hilbert space of asymptotic states has to be constructed. Such a space is usually referred in physical literature as a "Fock space".

The Hamiltonian coming out of this action is not Hermitian in the Fock space. This is solved with the identification above. The rest is a straightforward excercise. In the end we obtain the correct Lorentzian signature action that you will find in textbooks. back