# Lecture II

## Almost commutative geometry and the standard model

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Noncommutative Geometry and Applications to Quantum Physics

# Action!

We want to apply the ideas of the previous lecture to a quantum filed theory action, and later as a tool to describe the standard model of particle interaction.

The math we introduced are clearly geared for such a task. We already have (and this is of course no coincidence) the ingredients to write down the classical action of a filed theory

**Disclaimer!** I will be in an Euclidean and Compact spacetime. This is of course unphysical, but is a standard way to describe a field theory which solves several technical issues. Compactification is usually a mere technical device. The issue of Minkowski vs. Euclidean is more subtle. We will get back to it.

**Ex** Find at least two good reasons for which the construction of last lecture cannot be performed without changes for a noncompact and Minkowkian spacetime.

We have ready the matter fields. These are elements of the Hilbert space on which we represent the algebra (possibly with a reducible representation), and we will identify its elements with the fermionic matter fields of the theory

We should take into account the fact that the Dirac operator may fluctuate by the addition of a one form, a hermitean potential

$$\boxed{D_A = D + A}$$

Later we will refine the operator taking the presence of antiparticles into account

**Ex** Consider the toy model for which $\boxed{\mathcal{A}}$ is $\boxed{\mathbb{C}^2}$ represented as diagonal matrices on $\boxed{\mathcal{H} = \mathbb{C}^m \oplus \mathbb{C}^n}$, and $\boxed{D = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix}}$. Find $\boxed{D_A}$

The spectral action contains two part, one is the bosonic action:

$$S_B = \operatorname{Tr} \chi\left(\frac{D_A}{\Lambda}\right)$$

$\chi$ is the characteristic function of the interval $[0,1]$, or some smoothened version of it, and $\Lambda$ is a cutoff

This action must be read in a renormalization scheme. Namely we must insert it in a path integral and read the action coming from it. It will be later expanded with heath kernel techniques

For fermions we start introducing somewhat simplified version of the full one (again not taking into account charge conjugation)

$$S_F = \langle\Psi|D_A\Psi\rangle$$

The bosonic action is finite by construction, the fermionic part needs to be regularized

Consider the fermionic action alone, a theory in which fermions move in a fixed background

The classical action is invariant for Weyl rescaling

$$g^{\mu\nu} \to \mathrm{e}^{2\phi} g^{\mu\nu}$$

$$\psi \to \mathrm{e}^{-\frac{3}{2}\phi} \psi$$

$$D \to \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}$$

This is a symmetry of the classical action, not of the quantum partition function

$$Z(D) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi}$$

and therefore there is an anomaly because a classical symmetry is not preserved at the quantum level by a regularized measure.

We can therefore either "correct" the action to have an invariant theory, or consider a theory in which the symmetry is explicitly broken by a physical scale

In fact we need a scale to regularize the theory. The expression of the partition function can be formally written as a determinant:

$$Z(D,\mu) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi} = \det\left(\frac{D}{\mu}\right)$$

The determinant is still infinite and we need to introduce a cutoff

The regularization can be done in several ways. In the spirit of noncommutative geometry the most natural one is a truncation of the spectrum of the Dirac operator. This was considered long ago by Andrianov, Bonora, Fujikawa, Novozhilov, Vassilevich

The cutoff is enforced considering only the first $N$ eigenvalues of $D$

Consider the projector $P_N = \sum_{n=0}^{N} |\lambda_n\rangle \langle\lambda_n|$ with $\lambda_n$ and $|\lambda_n\rangle$ the eigenvalues and eigenvectors of $D$

$N$ is a function of the cutoff defined as $N = \max n \text{ such that } \lambda_n \leq \Lambda$

We effectively use the $N^{\text{th}}$ eigenvalue as cutoff

The choice of a sharp cutoff could be changed in favour of a cutoff function, similar to the choice of $\chi$

Define the regularized partition function

$$Z(D,\mu) = \prod_{n=1}^{N} \frac{\lambda_n}{\mu} = \det\left(\mathbf{1} - P_N + P_N \frac{D}{\mu} P_N\right)$$

$$= \det\left(\mathbf{1} - P_N + P_N \frac{D}{\Lambda} P_N\right) \det\left(\mathbf{1} - P_N + \frac{\Lambda}{\mu} P_N\right)$$

$$= Z_\Lambda(D,\Lambda) \det\left(\mathbf{1} - P_N + \frac{\Lambda}{\mu} P_N\right)$$

The cutoff $\Lambda$ can be given the physical meaning of the energy in which the effective theory has a phase transition, or at any rate an energy in which the symmetries of the theory are fundamentally different (unification scale)

The quantity $\mu$ in principle different and is a normalization scale, the one which changes with the renormalization flow

Under the change $\boxed{\mu \to \gamma\mu}$ the partition function changes

$$\boxed{Z(D,\mu) \to Z(D,\mu) e^{\frac{1}{\gamma} \operatorname{tr} P_N}}$$

On the other side

$$\boxed{\operatorname{tr} P_N = N = \operatorname{tr} \chi\left(\frac{D}{\Lambda}\right) = S_B(\Lambda, D)}$$

for the choice of $\boxed{\chi}$ the characteristic function on the interval, a consequence of our sharp cutoff on the eigenvalues.

We found the spectral action.

We could have started without it and the renormalization flow would have provided it for free.

Yet another way to find the bosnico action is to use the renormalization flow and $\boxed{\zeta}$ regularization. It is done in work with Kurkov, Sakellariadou and Watcharangkool

## Standard model of particle interaction

Presently we have a very successful (too successful?) model which interprets the interaction of the most elementary (at present) particles.

A particularly simple form of noncommutative geometry describes the standard model of particle interaction, the model investigated at CERN

The noncommutative geometry is particularly simple because it is the product of an infinite dimensional commutative algebra times a noncommutative finite dimensional one

Hence this algebra, being Morita equivalent of the commutative one describes a mild generalization of the space

The infinite dimensional part is the one relative to the four dimensional spacetime. I assume spacetime to be compact and Euclidean, which is not the case in the "real" world. I am not alone in making this assumption

We start from the algebra, a tensor productof the continuous function over a spacetime $M$ $\mathcal{A} = C(M) \otimes \mathcal{A}_F$, with the finite $\mathcal{A}_F = \text{Mat}(3, \mathbb{C}) \oplus \mathbb{H} \oplus \mathbb{C}$

$\mathbb{H}$ are quaternions

The unitaries of the algebra correspond to the symmetries of the standard model: $SU(3) \oplus SU(2) \oplus U(1)$

A unimodularity condition takes care of the extra U(1)

This algebra must be represented as operators on a Hilbert space, a continuos infinite dimensional part (spinors on spacetime) times a finite dimensional one: $\mathcal{H} = \text{sp}(\mathbb{R}) \otimes \mathcal{H}_F$.

Spinors (Euclidean) come with a chirality matrix (usually called $\gamma_5$ and charge conjugation, which associates to a spinor its (independent) hermitean conjugate. For the finite part we have a matrix $\gamma$ and an internal charge conjugation which is a matrix times complex conjugation

This algebra must be represented as operators on a Hilbert space, which also has a continuos infinite dimensional part (spinors on spacetime) times a finite dimensional one: $\mathcal{H} = \mathrm{sp}(M) \otimes \mathcal{H}_F$. The grading given by $\Gamma = \gamma_5 \otimes \gamma$ splits it into a left and right subspace: $\mathcal{H}_L \oplus \mathcal{H}_R$

The $J$ operator basically exchange the two chiralities and conjugates, thus effectively making the algebra act form the right.

As Hilbert space it is natural to take usual zoo of elementary particle, which transform as multiplets of standard model gauge group $SU(3) \times SU(2) \times U(1)$ always in the fundamental or trivial representation of the nonabelian groups. And under $U(1)$ their representation is identified by the weak hypercharge $Y$ (which has a factor of three in the normalization)

| Particle | $u_R$ | $d_R$ | $\begin{pmatrix} u_L \\ d_L \end{pmatrix}$ | $e_R$ | $\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ | $\nu_R$ |
|---|---|---|---|---|---|---|
| $SU(3)$ | 3 | 3 | 3 | 0 | 0 | 0 |
| $SU(2)$ | 0 | 0 | 2 | 0 | 2 | 0 |
| $Y$ | $\frac{4}{3}$ | $-\frac{2}{3}$ | $\frac{1}{3}$ | -2 | -1 | 0 |

Antiparticles have hyper charge reversed and left-exchanged with right

There are 2 kind of quarks, each coming in 3 colours, and 2 leptons, this makes 8. Times 2 (eigenspaces of the chirality $\gamma$). Times 2 with their antiparticles, eigenspaces of $J$.

Total: $\boxed{32}$.

Then the set of particle is repeated identically for three "generations". *Who ordered these?*

Grand Total: $\boxed{32 \times 3 = 96}$.

Note that there is some overcounting, actually a quadruplication of states

On one side $\boxed{\mathcal{H}_F}$ contains all of the particles degrees of freedom, including for example the electrons, right and left handed, and its antipaticles, left and right positrons. On the other we take the tensor product by a Dirac spinor, whose degrees of freedom are precisely electron and positron in two chiralities. It was because of this that Dirac predicted antimatter!

This quadruplication is actually historically called fermion doubling because it allows fermions of mixed chirality, which happens also in lattice gauge theory where there is just a duplication.

These uncertain chirality states are unphysical, and must be projected out, they are not just overcounting. But the projection must be done only in the action, not at the level of the Hilbert space.

We will come back to this doubling/quadruplication in the last lecture.

The algebra $\mathcal{A}_F$ should be represented on $\mathcal{H}_F$. $\mathbb{H}$ must act only on right-handed particles, $\mathrm{Mat}(3, \mathbb{C})$ only on quarks, . . .

Moreover, we need to satisfy the various constraints of NCG, the algebra commutes with the chirality, $[a, JbJ^\dagger = 0]$, . . .

I will not give the explicit expression of the representation. Upon request I will show it, but I want to convey the message that the fact that we have such a representation is quite "lucky". Very few gauge theory can have such a representation, but the standard model can.

An important aspect is that the representations on particles and antiparticles are different. Symmetry is restored acting on the opposite algebra $J\mathcal{A}J^\dagger$. The real structure is therefore fundamental.

I still have to give $D$. It will carry the metric information on the continuous part as well as the internal part.

For almost commutative geometries it splits into continuous and finite parts

$$D = \gamma^{\mu}(\partial_{\mu} + \omega_{\mu}) \otimes \mathbb{1} + \gamma^{5} \otimes D_F$$

$\omega_{\mu}$ the spin connection. We are in a curved background. The presence of $\gamma^5$, the chirality operator for the continuous manifold is for technical reasons.

All of the properties of the internal part are encoded in $D_F$, which is a $96 \times 96$ matrix.

# Fermionic action, masses

The presence of $\boxed{J}$ and its role for the representation of the algebra impose that the fluctuations of the Dirac operator must be

$$\boxed{D_A = D + A + JAJ^\dagger}$$

for a generic one-form A

Likewise the fermionic action must be written as

$$\boxed{S_F = \langle J\Psi | D_A \Psi \rangle}$$

Inserting in the proper entries of the matric $\boxed{D_F}$ the masses (Yukawa couplings) of the particles, gives the proper mass terms. The fluctuations of $\boxed{D}$ in the "internal" discrete space give another field, which we identify with the Higgs field

# Bosonic Action

$$S_B = \mathrm{tr}\,\chi\left(\frac{D_A}{\Lambda}\right) = \mathrm{tr}\,\chi\left(\frac{D_A^2}{\Lambda^2}\right)$$

for $\chi$ a step function. Otherwise the two functions are slightly different

We want to express this action in terms of the potential one-form, its curvature, the spin connection, Riemann tensors etc.

The standard technique used in that of the heath kernel

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of $\Lambda^{-1}$ as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2)$$

where the $f_n$ are the momenta of $\chi$

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\ f_{2n+4} &= (-1)^n \partial_x^n \chi(x) \Big|_{x=0} \quad n \geq 0 \end{aligned}$$

the $a_n$ are the Seeley-de Witt coefficients which vanish for $n$ odd. For $D^2$ of the form

$$D^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{1} + \alpha^\mu \partial_\mu + \beta)$$

Defining (in term of a generalized spin connection containing also the gauge fields)

$$\begin{aligned}\omega_\mu &= \frac{1}{2} g_{\mu\nu}\left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\mathbb{1}\right)\\ \Omega_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]\\ E &= \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right)\end{aligned}$$

then

$$\begin{aligned}a_0 &= \frac{\Lambda^4}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,\mathbb{1}_F\\ a_2 &= \frac{\Lambda^2}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,\left(-\frac{R}{6}+E\right)\\ a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu}\\ &\quad +2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu}\end{aligned}$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in $\Omega$ and $E$ are contained the gauge degrees of freedom including the gauge stress energy tensors and the Higgs, which is given by the inner fluctuations of $D$

**Now we can turn to physics, and use this framework to evaluate experimentally verifiable quantities**

**What are we going to predict?**

. . .

# References

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