Lecture I

Noncommutative spaces, physical origins, mathematical foundations

Fedele Lizzi

Università di Napoli *Federico II*

and Institut de Ciencies del Cosmos, Universitat de Barcelona

Noncommutative Geometry and Applications to Quantum Physics

Disclaimer

These lectures are given using a physicists point of view. I will attempt to give, within the restrictions of time, proper definitions of all the tools I will need, and prove (or give references) to the results I will need, but I will certainly lack the precision of my colleagues.

I will instead have with the rigour of a physicist. Namely, a "theory" is well founded if it help describe the nature around us, if it gives predictions which can be tested in experiments or observations. If it help us understand better the natural world

The plan of the lectures is the following

In today's lecture I will introduce noncommutative geometry from the spectral point of view, not only to set the stage for further developments, but to indicate the philosophy which will guide the course

After me Patrizia Vitale will further discuss, in her three lectures, differential noncommutative geometry and gauge theories on noncommutative spaces. The two set of lectures represent two sub-points of view, roughly speaking the views of Alain Connes on one side, and the one of the late Julius Wess form the other, with a sprinkle of Dubois-Violette.

For the remaining two lectures the geometry will be simpler, an ordinary spacetime multiplied by finite matrices, but I will apply the spectral point of view to the standard model of particle physics, bringing you dangerously close to the experiments currently done at LHC under the swiss alps. These lectures will give few details of the constructions. I will instead try to convey the flavours of the applications, leaving aside some details if (as it will be likely) I will will be short of time

Since the three lectures are of different length I may start the second one today.

Geometry has been always tied closely to physics. I like to offer a quote by Galileo Galilei from the Dialogo sui massimi sistemi

È forza confessare che il voler trattare le quistioni naturali senza geometria è un tentar di fare quello che è impossibile ad esser fatto

We must confess that to treat matters of nature without geometry is to attempt to do what cannot be done.

But what is Geometry?

to define it I cannot think of a higher authority than *Wikipedia*

Geometry (from the Ancient Greek: $\gamma\varepsilon\omega\mu\varepsilon\tau\rho\iota\alpha$ geo- "earth", -metron "measurement") is a branch of mathematics concerned with questions of shape, size, relative position of figures, and the properties of space.

Several physical theories do not just use geometry as a tool, they are geometry

Think for example of analytic mechanics, which is nothing but the symplectic geometry of phase space, or general relativity and Riemannian geometry

But the geometric view of mechanics had a crisis with the advent of the quantum world with the uncertainty principle

All at once the notion of a point in phase space becomes untenable, and the geometry made of points need to be generalised to describe the quantum phase space

The algebra of commuting functions on phase space become an algebra quantum noncommuting operators. In both cases the state of a physical system is a map from the element of the algebra into numbers

Commutative geometry

Hausdorff topological spaces as commutative algebras

There are two important series of theorems mostly due to Gelfand, Naimark and Segal which connect C^*-algebras and Hausdorff topological spaces.

A C^*-algebra $\mathcal{A}$ is an associative, normed algebra over a field (typically the complex $\mathbb{C}$), with an involution * which satisfies the following properties:

$$\|a+b\| \leq \|a\| + \|b\| \ , \ \|\alpha a\| = |\alpha| \ \|a\| \ , \|ab\| \leq \|a\| \ , \|b\|\|a\| \geq 0,$$

$$\|a\| = 0 \iff a = 0 \ , \|a^*\| = \|a\| \ , \quad \|a^* a\| = \|a\|^2 \ , \quad \forall \ a \in \mathcal{A}$$

A C^*-algebra is complete in the topology given by the norm, otherwise we talk of a $*$-algebra.

Examples are $\mathrm{Mat}(n, \mathbb{C})$, i.e. $n \times n$ matrices, or the algebra of bounded or compact operators on a Hilbert space, or the algebra $\mathcal{C}_0(M)$ of continuous functions over a Hausdorff spaces M vanishing on the frontier in the non-compact case, with pointwise multiplication. In the first case involution is Hermitean conjugacy, in the second complex conjugation of the function.

One has to be careful with the norm. Obvious norms like $\text{Tr}\, A^\dagger A, \quad A \in \text{Mat}(n, \mathbb{C})$ or $\int_M d\mu |f|^2, \quad , f \in \mathcal{C}_0(M)$, which come form a Hilbert space structure, do not satisfy the norm property for a $*$-algebra

Proper norms in the two cases are

$$\|A\|^2 = \max_{\text{eigenvalues}} A^\dagger A$$

$$\|f\|^2 = \sup_{x \in M} |f(x)|^2$$

In particular the second case is relevant. Given a Hausdorff space m it is always possible to define canonically a commutative C^*-algebra as that of continuous complex valued functions. Only if M is compact the algebra will be unital (will contain the identity).

A key result is that the inverse is also true:

A commutative C^*-algebra is the algebra of continuous complex valued functions on a Hausdorff space

In other words, Hausdorff spaces and C^*-algebra are in a one-to-one correspondence

The proof is constructive, given a C^*-algebra it is possible to reconstruct the points of the Hausdorff space, and their topology

For this we need the notion of state ϕ, i.e. a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ positive $\phi(a^* a) \leq 0$ and of norm one $\|\phi\| = \sup_{\|a\| \leq 1} \phi(a) = 1$

If the algebra is unital then it must be $\phi(1) = 1$

The space of states is convex, any linear combination of states of the kind $\cos^2\lambda\phi + \sin^2\lambda\phi$ is still a state for any value of λ.

Some states cannot be expressed as such convex sum, they form the boundary of the set and are called pure states.

Ex Find the states (and the pure ones) for the algebra of $n \times n$ matrices.

For a commutative algebra the pure states coincide with the (necessarily one-dimensional) irreducible representations, as well as the set of maximal and prime ideals. In the noncommutative case these sets are different.

Gelfand and Naimark gave a prescription to reconstruct a topological space in an unique way from a commutative algebra

The topology on the space of pure states is given by defining the limit. Given a succession of pure states δ_{x_n} define the limit to be

$$\lim_n \delta_{x_n} = \delta_x \Leftrightarrow \lim_n \delta_{x_n}(a) = \delta_x(a) \ , \forall a \in \mathcal{A}$$

In other words the states which correspond to the points are the evaluation maps whereby the complex number associated to a function is simply the value of the function in the point

With the above topology the starting algebra results automatically the algebra of continuous functions over the space of states

Similarly it is possible to give a topology on the set of ideals, leading to the same topology for the commutative case.

Algebras as operators: the GNS construction

As I said NCG is a spectral point of view. This is a consequence of the fact that any C^*-algebra is always representable as bounded operators on a Hilbert space. The proof is again constructive, and is called the Gelfand-Naimark-Segal (GNS) construction

Every algebra has an action on itself. Consider the algebra itself as the starting vector space for the construction of the Hilbert space. We then need an inner product with certain properties, and then we need to complete in the norm given by this product.

Any state ϕ gives a bilinear map $\langle a, b\rangle = \text{Tr}\, a^*b$ with some of the properties of the inner product. The problem is that there are some elements of the lgerba for which $\text{Tr}\, a^*a$ is be zero, even if a is not the null vector. Those states form a (left) ideal $\mathcal{N}_\phi$ and have to quotientied out

Ex Prove that $\mathcal{N}_\phi$ is an ideal

Consider now the vector space $[a] \in \mathcal{A}/\mathcal{N}_\phi$. The scalar product

$$\langle [a], [b] \rangle = \phi(a^* b)$$

is well defined since it is easy to see that is independents on the representatives a and b in the equivalence classes

We have this given a scalar product which defines a good norm. The Hilbert space is defined completing in the scala product norm, on which $\mathcal{A}$ acts in a natural way as bounded operators.

Ex Given the algebra $\mathcal{C}_0(\mathbb{R})$ consider the two states $\delta_{x_0}(a) = a(x_0)$ and $\phi(a) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \, e^{-x^2} a(x)$. Find the Hilbert space in the two cases.

Ex Find the pure states and perform the GNS construction for $\mathrm{Mat}(n, \mathbb{C})$

Noncommutative spaces. Morita equivalence

Once established the correspondence between Hausdorff spaces and commutative C^*-algebras we may ask what happens for noncommutative algebras

It is still possible to give topologies on the spaces of irreducible representations (not anymore necessarily one-dimensional) , pure states and maximal or primitive ideals, but these do not coincide anymore.

For us a noncomutative space will be, by extension of the concept, a noncommutative C^*-algebra, sometimes it will possible to talk of point, possibly generalized, other times it simply does not make sense.

Think for example of the algebra of quantum operators generated by $\hat{p}$ and $\hat{q}$ of quantum mechanics. This is a deformation of the ordinary algebra of functions on phase space. Pure states are square integrable functions on $\mathbb{R}$ which are in no correspondence with the original points. Coherent states are the closest to the concept of point, but are in no way "pointlike".

Consider instead the case of matrix valued functions on $\mathbb{R}$. The algebra is noncommutative and obviously there is an underlying space, $\mathbb{R}$ itself!

This fact is captured by the concept of Morita Equivalence

We first need the concept of Hilbert Module. This is a generalization of Hilbert spaces where the field $\mathbb{C}$ is replaced by a C^*-algebra. A right Hilbert module $\mathcal{E}$ over $\mathcal{A}$ is a right module equipped with a sesqulinear form $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$ linear in the first variable and with

$$\langle \eta_1 | \eta_2 a \rangle_{\mathcal{A}} = \langle \eta_1 | \eta_2 \rangle_{\mathcal{A}} a \ , \ \langle \eta_1 | \eta_2 \rangle_{\mathcal{A}}^* = \langle \eta_2 | \eta_1 \rangle_{\mathcal{A}} \ ,$$

$$\langle \eta | \eta \rangle_{\mathcal{A}} \geq 0 \ , \quad \langle \eta | \eta \rangle_{\mathcal{A}} = 0 \ \Leftrightarrow \ \eta = 0$$

$\forall \eta_1, \eta_2, \eta \in \mathcal{E}, \ a \in \mathcal{A}$

Completion in the norm given $\|\eta\|_{\mathcal{A}}^2 = \| \langle \eta | \eta \rangle \|_{\mathcal{A}}$ is assumed.

A left Hilbert module is defined in an analogous way.

Ex Make $\mathcal{A}^N$ into a Hilbert module, and discuss its automorphisms

Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be Morita equivalent if there exists a full right Hilbert $\mathcal{A}$-module $\mathcal{E}$ which is at the same time a left $\mathcal{B}$-module in such a way that the structures are compatible

$$\langle\eta|\,\xi\rangle_{\mathcal{B}}\,\zeta = \eta\,\langle\xi|\,\zeta\rangle_{\mathcal{A}}\ , \quad \forall\ \eta,\xi,\zeta \in \mathcal{E}$$

Ex The algebras $\mathbb{C}, \mathrm{Mat}(\mathbb{C}, n)$ and compact operators on a Hilbert space are all Morita equivalent. Find the respective bimodules

Two Morita equivalent algebras have the same space of representations, with the same topology. They describe the same noncommutative space.

The space of complex valued, or matrix valued functions, since a matrix algebra has only one representation, are Morita equivalent.

But Morita equivalence is far from being “physical” equivalence. There is more than topology!

Beyond topology: metric aspects

Let us now introduce one of the main ingredients for the Connes vision of noncommutative geometry. It adds to the algebra $\mathcal{A}$ which acts as bounded operators on Hilbert space $\mathcal{H}$.

It is a self-adjoint operator D on $\mathcal{H}$ with compact resolvent. It can be seen as a generalization of the Dirac operator. I will call it Dirac operator, even of oftentimes it will not look at all as the operator introduced by Paul Dirac ninety years ago.

Together $\mathcal{A}, D, \mathcal{H}$ form what is called a spectral triple

The Dirac operator enables to describe, in purely algebraic operatorial terms, the usual structures of geometry. Since the description is purely algebraic it can easily be generalized to the noncomutative case, thus enabling a non-commutative geometry.

The presence of D enables for example to give a distance, and hence a metric structure, on the space of states on an algebra

$$d(\phi_1, \phi_2) = \sup_{\|[D,a]\| \leq 1} \{|\phi_1(a) - \phi_2(a)|\}$$

Ex Take $\mathcal{A} = \mathcal{C}(\mathbb{R})$ and $D = \partial_x$. Prove that the distance among pure states gives the usual distance among points of the line $d(x_1, x_2) = |x_1 - x_2|$

A comment for physicsts, the original Dirac operator $\gamma^\mu \partial_\mu$ "knows" a lot about a spin manifold. The differential structure,the spin structure, but also the metric tensor since $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$, it is the square root of the laplacian. It is therefore no surprise that it plays such an important role in pure mathematics.

The presence of D enables an algebraic definition of one-forms and the creation of a cohomology via $da \sim [D, a]$. A generic one form will be $A = \sum_i a_i [D, b_i]$

The construction is delicate, one has to implement $d^2 a = 0$. To achieve this one has to quotient out some "junk" forms. I defer for details to the books of Landi or Gracia-Bondìa, Varilly and Figueroa for example.

The dimension can be obtained form the rate of growth of the eigenvalues of D^2 (Weyl). Consider the ratio of number N_ω of eigenvalues smaller than a value ω. Then

$$\lim_{\omega \to \infty} \frac{N_\omega}{\omega^{\frac{d}{2}}}$$

Does not diverge or vanishes for a single value of d, which defines the dimension, and is of course the usual Hausdorff dimension for the case of manifolds and the usual D

Integrals are substituted by trace, in particular the Diximier trace which is mentioned elsewhere in this school

The message I wish to convey is that a Dictionary is being built, whereby the concepts of differential are translated in a purely algebraic way

In this way they can be generalized to the cases in which the algebra is noncommutative

This procedure must work also in the cases which are not deformation of ordinary geometries

Noncommutative manifolds

Manifolds play a central role in geometry, but how do we translate their differential structure?

We need requirements which, when applied to the commutative case characterize manifolds, and which can be generalised. To do this we need to add two more ingredients to the spectral triple, they are both operators on $\mathcal{H}$

Interestingly, both play an important role in quantum field theory

The Chirality Operator Γ with $\Gamma^2 = \mathbb{1}$ actually exists only in the even case. It splits $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$

A "charge conjugation" antiunitary operator J which gives a real structure.

It is connected to the Tomita-Takesaki operator, in case you know what this is

Connes has shown that the following seven "axioms" characterize manifolds in the commutative case, and generalize to the noncommutative one

1. Dimension This has been discussed above

2. Regularity For any $a \in \mathcal{A}$ both a and $[D, a]$ belong to the domain of δ^k for any integer k, where δ is the derivation given by $\delta(T) = [|D|, T]$.

3. Finiteness The space $\bigcap_k \mathrm{Dom}(D^k)$ is a finitely generated projective left $\mathcal{A}$ module.

4. **Reality There exist J with the commutation relation fixed by the number of dimensions with the property**

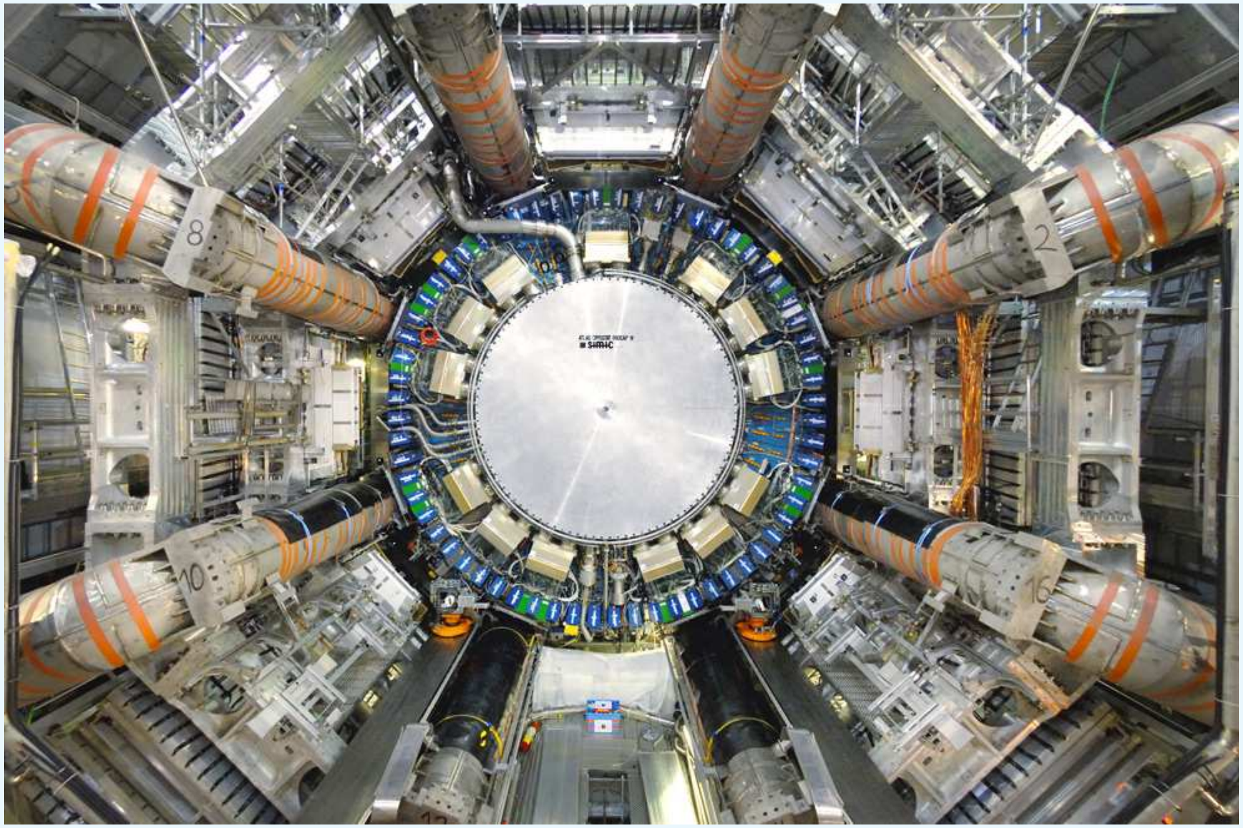
   (a) **Commutant** $[a, Jb^* J^{-1}] = 0, \forall a, b$

   (b) **First order** $[[D, a], b^o = Jb^* J^{-1}] = 0 \ , \forall a, b$

5. Orientation There exists a Hochschild cycle c of degree n which gives the grading γ , This condition gives an abstract volume form.

6. Poincaré duality A Certain intersection form determined by D and by the K-theory of $\mathcal{A}$ and its opposite is nondegenerate.

So what has this to do with the
Large Hadron Collider ?



To be continued. . .

References

- Any lecture on NCG should have as first reference the book: A. Connes Noncommutative Geometry Wiley 1994

- A more modern work by the master is: A. Connes, M. Marcolli Noncommutative Geometry, Quantum Fields and Motives AMS 2007

- A complete mathematical compendium is: J.M. Gracia-Bondia, J. Varilly, H. Figueroa Elements of Noncommutative Geometry Birkhauser 2000

- Another excellent intro: G. Landi An Introduction to Noncommutative Spaces and their Geometries Springer 1997

- For A short version of what discussed in this lecture there is chap 6 of.: P. Aschieri, M. Dimitrijevic, P- Kulish, F. Lizzi, J. Wess Noncommutative spacetimes: symmetries in noncommutative geometry and field theory Academic Press 1988

- For the topology of the space of representation, GNS etc.: J.M.G. Doran, R.S. Fell Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles Academic Press 1988

- For the Characterization of the manifold: A. Connesl On the spectral characterization of manifolds Alain Connes J. Noncommut. Geom. 7 (2013), 1; arXiv:0810.2088