Non-Commutative Topology and Topological Quantisation

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III K-theory for insulators

Recapitulation

$\mathcal{E}$ is the collection of $\mathbb{R}$-linear maps expressing the extra-ordinary symmetries of insulators (time reversal, charge conjug., chiral sym.).

The group classifying the extended topological phases of insulators with algebra A and symmetry $\mathcal{E}$ is

$$DK_e(A, \mathcal{E}) := \mathfrak{G} \varinjlim {}^e \mathcal{GL}_m(A)^{\{*, \mathcal{E}\}} / \sim_h$$

The higher K-groups of a graded C^*-algebra (B, γ) are

$$K_i(B, \gamma) := DK_e(B \hat{\otimes} Cl_{r,s}, \{-\gamma \otimes \mathrm{st}\}), \quad r - s = 1 - i$$

(If B unbalanced must have $r > 0$ or $s > 0$)

The elements are differences of homotopy classes of OSUs (odd self-adjoint unitaries).

Customary notation: $KU_i(A) = K_i(A)$ for trivially graded complex, $KO_i(A) = K_i(A)$ for trivially graded real A.

Let A be trivially graded.

$K_0(A) = DK_{1\otimes\rho}(A \otimes Cl_{1,0})$. Any OSU has form $x = h \otimes \rho$ with $h = h^* = h^{-1}$, so $p = \frac{1-h}{2} \in Proj_m(A)$
$\mathcal{U}_m(A \otimes Cl_{1,0})^{\{*,-\mathrm{st}\}} \ni h \otimes \rho \mapsto \frac{1-h}{2} \in Proj_m(A)$ induces

$$DK_{1\otimes\rho}(A \otimes Cl_{1,0}) \cong \mathfrak{G} \varinjlim{}^0 \mathcal{U}_m(A)^{\{*\}} / \sim_h \cong \mathfrak{G} \varinjlim{}^0 Proj_m(A) / \sim_h$$

$K_1(A) = DK_{1\otimes\sigma_x}(A \otimes Cl_{1,1})$. Any OSU has form $x = u \otimes \sigma_x + v \otimes i\sigma_y$ with $u + v$ unitary
$\mathcal{U}_m(A \otimes Cl_{1,1})^{\{*,-\mathrm{st}\}} \ni u \otimes \sigma_x + v \otimes i\sigma_y \mapsto u + v \in U_m(A)$ induces

$$DK_{1\otimes\sigma_x}(A \otimes Cl_{1,1}) \cong \mathfrak{G} \varinjlim{}^1 \mathcal{U}_m(A) / \sim_h$$

III van Daele K-theory

Trivially graded algebras II

Let $(A,\mathfrak{r})$ be trivially graded complex algebra with real structure $\mathfrak{r}$.

$K_2(A^{\mathfrak{r}}) = DK_{Y\otimes i\rho}(A^{\mathfrak{r}} \otimes Cl_{0,1})$. Any OSU has form $x = h \otimes \rho$ with $h = h^* = h^{-1}$ and $\mathfrak{r}(h) = -h$, so $p = \frac{1-h}{2}$ satisfies $\mathfrak{r}(p) = p^{\perp}$
$\mathcal{U}_m(A \otimes Cl_{1,0})^{\{*,-\mathrm{st}\}} \ni h \otimes \rho \mapsto h \in \mathcal{U}_m(A)^{\{*\}}$ induces

$$DK_{1\otimes\rho}(A^{\mathfrak{r}} \otimes Cl_{1,0}) \cong \mathfrak{G}\varinjlim{}^{\sigma_y}\mathcal{U}_m(A)^{\{*,-\mathfrak{r}\}}/\sim_h$$

$K_{-1}(A^{\mathfrak{r}}) = DK_{1\otimes\sigma_x}(A^{\mathfrak{r}} \otimes Cl_{2,0})$. Any OSU has form $x = u \otimes \sigma_x + v \otimes \sigma_y$ with $u + iv \in M_m(A)$ unitary and $\mathfrak{r}(u) = u$, $\mathfrak{r}(v) = v$. Hence $\mathfrak{r}(u+iv) = u - iv = (u+iv)^*$
$\mathcal{U}_m(A \otimes Cl_{2,0})^{\{*,-\mathrm{st}\}} \ni u \otimes \sigma_x + v \otimes \sigma_y \mapsto u + iv \in U_m(A)^{\{\mathfrak{r}^*\}}$ induces

$$DK_{1\otimes\sigma_x}(A^{\mathfrak{r}} \otimes Cl_{2,0}) \cong \mathfrak{G}\varinjlim{}^{1}\mathcal{U}_m(A)^{\{\mathfrak{r}^*\}}/\sim_h$$

Boersema & Loring's picture of KU and KO-theory in terms of unitaries with extra structure

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If A is complex trivially graded then

$$KU_i(A) = \mathfrak{G} \lim_{\rightarrow} {}^e\mathcal{U}_m(A)^{\{\alpha\}} / \sim_h$$

where

i	0	1
$\{\alpha\}$	$\{*\}$	$\emptyset$
e	1	1

If $(A, \mathfrak{r})$ is complex trivially graded with real structure $\mathfrak{r}$ then

$$KO_i(A^{\mathfrak{r}}) = \mathfrak{G} \lim_{\rightarrow} {}^e\mathcal{U}_m(A)^{\{\alpha\}} / \sim_h$$

where

i	-1	0	1	2
$\{\alpha\}$	$\{\mathfrak{r}^*\}$	$\{*, \mathfrak{r}\}$	$\{\mathfrak{r}\}$	$\{*, -\mathfrak{r}\}$
e	1	1	1	σ_y

Insulator picture of KU and KO-theory.

Let A be the (complex) algebra associated to the insulator, $\mathcal{E}$ the set of its extra ordinary symmetries.
Its extended topological phases are classified by $DK_e(A, \mathcal{E})$

$DK_e(A, \mathcal{E})$	$KU_0(A)$	$KO_0(A^{\mathfrak{r}})$	$KO_2(A^{\mathfrak{r}})$
$\mathcal{E}$	$\emptyset$	$\{\mathfrak{r}\}$	$\{-\mathfrak{r}\}$

and, assuming that chiral symmetry is inner (with generator Γ)

$DK_e(A, \mathcal{E})$	$KO_1(A)$	$KO_1(A^{\mathfrak{r}})$	$KO_{-1}(A^{\mathfrak{r}})$
$\mathcal{E}$	$\{-\mathrm{Ad}_\Gamma\}$	$\{\mathfrak{r}, -\mathrm{Ad}_\Gamma\}, \mathfrak{r}(\Gamma) = \Gamma$	$\{\mathfrak{r}, -\mathrm{Ad}_\Gamma\}, \mathfrak{r}(\Gamma) = -\Gamma$

Cyclic cohomology for graded algebras

The cyclic cohomology of a graded algebra A is the cohomology of a complex

$$C^0_\lambda(A) \xrightarrow{b} C^1_\lambda(A) \xrightarrow{b} \cdots$$

where $C^n_\lambda(A)$ is the module of n-cocycles, the $n+1$-linear forms $\xi : A^{n+1} \to \mathbb{C}$ which are cyclic. b is the coboundary operator.

The cohomology groups are not important here, only the cocycles. These can be constructed as characters of cycles over A.

Cycles and their characters for graded algebras

Definition

A cycle $(\Omega, d, \int)$ of dimension n over a complex balanced graded algebra (A, γ, e) (choice of base points e) is

- a $\mathbb{Z} \times \mathbb{Z}_2$-graded differential algebra (Ω, d), d has degree $(1, 0)$
- a graded inclusion $A \hookrightarrow \Omega_0$ ($\mathbb{Z}$-degree 0)
- a closed $\mathbb{Z} \times \mathbb{Z}_2$-graded trace $\int : \Omega_n \to \mathbb{C}$
  - $\int d\omega = 0$
  - $\int \omega\omega' = (-1)^{|\omega|_{\mathbb{Z}}|\omega'|_{\mathbb{Z}} + |\omega|_{\mathbb{Z}_2}|\omega'|_{\mathbb{Z}_2}} \int \omega'\omega$
- $\exists$ OSU $e \in A$ s.th. $de = 0$.

The character ξ of the cycle is a cyclic n-cocycle

$$\xi(a_0, \cdots, a_n) = \int a_0 da_1 \cdots da_n$$

Cycles may only be densely defined (higher traces)

Extension of cycles to $A\hat{\otimes}\mathbb{C}l_k$

Let $\jmath_k : \mathbb{C}l_k \to \mathbb{C}$, $i_l < i_{l+1}$

$$\jmath_k(\rho_{i_1} \cdots \rho_{i_l}) = \begin{cases} 1 & \text{if } l = k \\ 0 & \text{if } l < k \end{cases}$$

- $(\mathbb{C}l_k, 0, \jmath_k)$ is a cycle of dimension 0 over $\mathbb{C}l_k$.
- $(\Omega\hat{\otimes}\mathbb{C}l_k, d \otimes 1, \int \circ \jmath_k)$ is a cycle of dimension n over $A\hat{\otimes}\mathbb{C}l_k$.

Pairing of cyclic cocycles with $DK_e(A\hat{\otimes}\mathbb{C}l_k, \gamma\otimes\mathrm{st})$

Definition

Let ξ be the character of an n-dim. cycle $(\Omega, d, \int)$ over a complex balanced graded C^*-alg. (A, γ, e). The pairing of $[x] \in DK_e(A\hat{\otimes}\mathbb{C}l_k)$ with ξ is

$$\langle \xi, [x]\rangle = 2^{\frac{k}{2}} \int \jmath_k \mathrm{Tr}_m(x - e_m)(dx)^n \qquad (1)$$

$x \in \mathcal{U}_m(A\hat{\otimes}\mathbb{C}l_k)^{\{*,-\gamma\otimes\mathrm{st}\}}$, $e_m = e \oplus \cdots \oplus e$.

- One cycle $(\Omega, d, \int)$ yields a pairing with all higher K-groups.
- For $(A, \mathfrak{r})$ (1) yields pairing with $DK_e(A^{\mathfrak{r}}\hat{\otimes}Cl_{r,s})$, $k = r + s$.
- If A trivially graded (1) is Connes' pairing with $KU_i(A)$, $i = 1 - k$.
- If $(A, \mathfrak{r})$ triv. graded (1) yields pairing with $KO_i(A^{\mathfrak{r}})$, $i = 1 - r + s$, $k = r + s$.

Standard cycles for $\mathbb{Z}^n$-actions

Let (A, γ) graded complex C^*-algebra. Suppose we have

- n commuting $*$-derivations ∂_i on A
- closed graded trace $\mathrm{Tr_A}$ on A

Then $(\Omega, d, \int)$ is n-dim cycle where

- $\Omega = A \otimes \Lambda\mathbb{C}^n$
  - $\Lambda\mathbb{C}^n$ Grassmann algebra with odd self-adj. generators λ_j
  - $\mathbb{Z}_2$-grad. $\gamma \otimes \mathrm{id}$, $\mathbb{Z}$-grad. from Λ
- $da = \sum_{i=1}^n \partial_i a \wedge \lambda_i$
- $\int = \mathrm{Tr_A} \circ \imath$,
  - $\imath : \Lambda^n\mathbb{C} \to \mathbb{C}$, $\imath(\lambda_1 \wedge \cdots \wedge \lambda_n) = 1$

Examples for standard cycles

- Let M an n-dim. closed orientable manifold. $(\Omega(M), d, \int_M)$ is a cycle over the trivially graded algebra $C(M)$.
  - $(\Omega(M), d)$ algebra of differential forms, d the exterior derivative
  - $\int_M : \Omega^n(M) \to \mathbb{R}$ integration of n-forms
- Standard n-cocyle ch_n for $A_{\mathcal{L}^\bullet,\phi} = C_{\mathcal{L}^\bullet}(\mathcal{L}) \rtimes_{\alpha,\phi} \mathbb{Z}^d$
  - ∂_i infinitesimal dual action: $\partial_i(T_\phi^{e_j}) = \delta_{ij} T_\phi^{e_j}$
  - $\mathrm{Tr}_{\mathrm{A}}(\sum_n H_n T_\phi^n) = \sum_{x \in \mathcal{L}} H_0(x)$.

  If $\mathcal{L}^\bullet$ is Λ-periodic then, intertwined with the Fourier transform, this example becomes the first one with $M = \mathbb{R}^d/\Lambda^{rec}$, extended by $\otimes M_K(\mathbb{C})$ where $K = |\mathcal{L}^\bullet/\Lambda|$.

If H is the Hamiltonian of the insulator then $\langle \mathrm{ch}_d, [H] \rangle$ is the strong topological invariant of H.

If H is Λ-periodic then $\langle \mathrm{ch}_d, [H] \rangle$ is the nth Chern number of the vector bundle over $M = \mathbb{R}^d/\Lambda^{rec}$ defined by the negative energy states.

Physical interpretation of strong topological inv.

Since ∂_i is a derivation, we have $\partial_i(T_\phi^n) = n_i T_\phi^n = [X_i, T_\phi^n]$ where $X_i\Psi(x) = x_i\Psi(x)$ is the position operator on $\ell^2(\mathcal{L})$ ($\mathcal{L}$ is a lattice). It follows that (for d even)

$$\langle \mathrm{ch}_d, [H] \rangle = c_n \sum_{\sigma \in Perm_n} sgn(\sigma)\mathcal{T}(\tilde{H}[X_{\sigma(1)}, \tilde{H}] \cdots [X_{\sigma(d)}, \tilde{H}])$$

$\tilde{H} = H|H|^{-1}$ is spectrally flattened Hamitlonian
$\mathcal{T}$ the trace per unit volume. c_n a normalisation.

- $i[X_i, H]$ is the velocity operator.
- In linear response theory to perturbations by other external fields (e.g. electric), in a relaxation time approximation, in a limit where relaxation time goes to infinity and temperature to 0 (carefully taken) the above is a d-point current current correlation function.
- Such correlation functions yield topologically quantised transport coefficients.
- In the Hofstadter model this is the transverse (Hall) conductivity.

Exercise

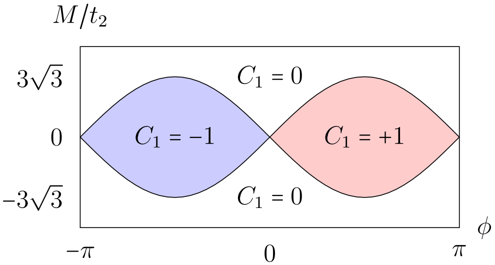
Let H be the Haldane Hamitonian (with gap shifted to 0).

- Show that the strong invariant is given by

$$\langle \mathrm{ch}_2, [H] \rangle = 8 \int_{\mathbb{R}^2/\Lambda^{rec}} \mathrm{Tr}\left(P_-(k) \left[\frac{\partial P_-(k)}{\partial k_1}, \frac{\partial P_-(k)}{\partial k_2} \right] \right)$$

  Here $P_-(k)$ is the projection onto the negative energy states of $\hat{H}(k) - d_0(k)1$.

- Compute the strong invariant for the 3 different domains separated by the curves in the diagram below.



Answer (up to banana-schnaps constants): $\frac{1}{16\pi i}\langle \mathrm{ch}_2, [H] \rangle = C_1$