Non-Commutative Topology and Topological Quantisation

Johannes Kellendonk

Institut Camille Jordan, Université Lyon 1

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III K-theory for insulators

Recall: The topological phase of the insulator is the homotopy class of H in $\mathcal{GL}(A)^{\{*,\mathcal{G}\}}$
$\mathcal{G}$ a collection of $\mathbb{R}$-linear maps expressing the symmetries. For $\alpha \in \mathcal{G}$, α or $-\alpha$ is an automorphism which is $\mathbb{C}$-linear or anti-linear. $\alpha(H) = H$.

- Homotopy is difficult to compute.
- Turn it into K-groups! (abelian groupisation)
  easier to compute, little loss of information

III K-theory for insulators

Construct a monoid (semigroup with neutral element) as follows:
(A is unital)

1. Extend the $\alpha \in \mathcal{G}$ entrywise to $M_m(A)$. Denote extension simply by α.
2. Let $e \in \mathcal{GL}(A)^{\{*,\mathcal{G}\}}$ (basepoint)
3. $M_m(A) \ni x \mapsto x \oplus e \in M_{m+1}(A)$ preserves $\alpha \in \mathcal{G}$ and $*$
4. The direct limit $\varinjlim^e \mathcal{GL}_m(A)^{\{*,\mathcal{G}\}} / \sim_h$ of homotopy classes is a semigroup under $[x] + [y] = [x \oplus y]$.
5. Applying the Grothendieck functor $\mathfrak{G}$ we obtain an abelian group

$$DK_e(A, \mathcal{G}) := \mathfrak{G} \varinjlim{}^e \mathcal{GL}_m(A)^{\{*,\mathcal{G}\}} / \sim_h$$

The neutral element is the class of e.

The extended topological phase of an insulator is the class of its Hamiltonian in $DK_e(A, \mathcal{G})$

III van Daele K-theory

- Split off the ordinary symmetries $\mathcal{G} = G \cup \mathcal{E}$.
- $\mathcal{GL}(A)^{\{*,\mathcal{G}\}} = \mathcal{GL}(A^{\mathcal{G}})^{\{*,\mathcal{E}\}}$ and we have 5 possibilities $\mathcal{E} = \emptyset, \{-\gamma\}, \{\mathfrak{r}\}, \{-\mathfrak{r}\}, \{-\gamma, \mathfrak{r}\}$.
- We assume from now on $A = A^G$ and organise the groups $DK_e(A, \mathcal{E})$ so that they become different degrees of the usual van Daele K-group.

III van Daele K-theory

Def. [van Daele '84] The van Daele group of a balanced graded (complex or real) C^*-algebra (A,γ) is

$$DK_e(A) := \mathfrak{G} \varinjlim{}^e \mathcal{U}_m(A)^{\{*,-\gamma\}} / \sim_h$$

- $\mathcal{U}_m(A)^{\{*,-\gamma\}} / \sim_h = \mathcal{GL}_m(A)^{\{*,-\gamma\}} / \sim_h$ (unitaries are a retract of invertibles)
- The elements of $DK_e(A,\gamma)$ are differences of homotopy classes of OSUs (odd self-adjoint unitary)
- Addition is abelian and $[e]$ is the neutral element.
- The map $x \mapsto x \oplus e$ induces an isomorphism $DK_e(A,\gamma) \cong DK_{e\oplus e}(M_2(A),\gamma_2)$ (stability of DK_e).
- $DK_e A,\gamma)$ depends on e up to isomorphism.
- DK_e is a functor from the category of graded unital C^*-algebras with graded unital $*$-morphisms to category of abelian groups.
- $DK_e(A,\gamma) \cong KK_1(\mathbb{C},A)$ (Kasparov K-theory for graded algebras)

For a balanced graded (complex or real) C^*-algebra (A, γ)

$$K_i(A, \gamma) := DK_e(A \hat{\otimes} Cl_{r,s}, \gamma \otimes \mathrm{st}), \quad r - s = 1 - i$$

- $(A \hat{\otimes} Cl_{1,1}, \gamma \otimes \mathrm{st},) \cong (M_2(A), \gamma_2)$ so well defined
- $A \hat{\otimes} Cl_{r,s} = A \hat{\otimes} \mathbb{C}l_{r+s}$ if A is complex
- $(A \hat{\otimes} Cl_{2,0}, \gamma \otimes \mathrm{st}) \cong (M_2(A), \gamma_2)$ if A cplx so $K_{i+2}(A^{\mathfrak{r}}, \gamma) \cong K_i(A^{\mathfrak{r}}, \gamma)$
- $(A \hat{\otimes} Cl_{8,0}, \gamma \otimes \mathrm{st}) \cong (M_{16}(A), \gamma_{16})$ so $K_{i+8}(A^{\mathfrak{r}}, \gamma) \cong K_i(A^{\mathfrak{r}}, \gamma)$
- $(A \hat{\otimes} Cl_{4,0}, \gamma \otimes \mathrm{st}) \cong (A \hat{\otimes} \mathbb{H}, \gamma \otimes \mathrm{id})$ so $K_{i+4}(A) \cong K_i(A \hat{\otimes} \mathbb{H})$

III van Daele K-theory

Trivially graded algebras 1

Customary notataion: $KU_i(A) = K_i(A)$ for trivially graded complex, $KO_i(A) = K_i(A)$ for trivially graded real A.

Let A be trivially graded.

$K_0(A) = DK_{1\otimes\rho}(A \otimes Cl_{1,0})$. Any OSU has form $x = h \otimes \rho$ with $h = h^* = h^{-1}$, so $p = \frac{1-h}{2} \in Proj_m(A)$

$\mathcal{U}_m(A \otimes Cl_{1,0})^{\{*,-\mathrm{st}\}} \ni h \otimes \rho \mapsto \frac{1-h}{2} \in Proj_m(A)$ induces

$$DK_{1\otimes\rho}(A \otimes Cl_{1,0}) \cong \mathfrak{G} \varinjlim{}^0 \mathcal{U}_m(A)^{\{*\}} / \sim_h \cong \mathfrak{G} \varinjlim{}^0 Proj_m(A) / \sim_h$$

$K_1(A) = DK_{1\otimes\sigma_x}(A \otimes Cl_{1,1})$. Any OSU has form $x = u \otimes \sigma_x + v \otimes i\sigma_y$ with $u + v$ unitary

$\mathcal{U}_m(A \otimes Cl_{1,1})^{\{*,-\mathrm{st}\}} \ni u \otimes \sigma_x + v \otimes i\sigma_y \mapsto u + v \in U_m(A)$ induces

$$DK_{1\otimes\sigma_x}(A \otimes Cl_{1,1}) \cong \mathfrak{G} \varinjlim{}^1 \mathcal{U}_m(A) / \sim_h$$

III van Daele K-theory

Trivially graded algebras II

Let $(A, \mathfrak{r})$ be trivially graded complex algebra with real structure.

$K_2(A^{\mathfrak{r}}) = DK_{Y \otimes i\rho}(A^{\mathfrak{r}} \otimes Cl_{0,1})$. Any OSU has form $x = h \otimes \rho$ with $h = h^* = h^{-1}$ and $\mathfrak{r}(h) = -h$, so $p = \frac{1-h}{2}$ satisfies $\mathfrak{r}(p) = p^{\perp}$
$\mathcal{U}_m(A \otimes Cl_{1,0})^{\{*,-\mathrm{st}\}} \ni h \otimes \rho \mapsto h \in \mathcal{U}_m(A)^{\{*\}}$ induces

$$DK_{1 \otimes \rho}(A \otimes Cl_{1,0}) \cong \mathfrak{G} \varinjlim^{\sigma_y} \mathcal{U}_m(A)^{\{*,-\mathfrak{r}\}} / \sim_h$$

$K_{-1}(A^{\mathfrak{r}}) = DK_{1 \otimes \sigma_x}(A^{\mathfrak{r}} \otimes Cl_{2,0})$. Any OSU has form $x = u \otimes \sigma_x + v \otimes \sigma_y$ with $u + iv \in M_m(A)$ unitary and $\mathfrak{r}(u) = u$, $\mathfrak{r}(v) = v$. Hence $\mathfrak{r}(u + iv) = u - iv = (u + iv)^*$
$\mathcal{U}_m(A \otimes Cl_{2,0})^{\{*,-\mathrm{st}\}} \ni u \otimes \sigma_x + v \otimes \sigma_y \mapsto u + iv \in U_m(A)^{\{\mathfrak{r}^*\}}$ induces

$$DK_{1 \otimes \sigma_x}(A \otimes Cl_{2,0}) \cong \mathfrak{G} \varinjlim^{1} \mathcal{U}_m(A)^{\{\mathfrak{r}^*\}} / \sim_h$$

Boersema & Loring's picture of *KU* and *KO*-theory

If A is complex trivially graded then

$$KU_i(A) = \mathfrak{G}\lim_{\rightarrow} {}^e\mathcal{U}_m(A)^{\{\alpha\}} / \sim_h$$

where

i	0	1
$\{\alpha\}$	$\{*\}$	$\emptyset$
e	1	1

If A is complex trivially graded with real structure $\mathfrak{r}$ then

$$KO_i(A^{\mathfrak{r}}) = \mathfrak{G}\lim_{\rightarrow} {}^e\mathcal{U}_m(A)^{\{\alpha\}} / \sim_h$$

where

i	-1	0	1	2
$\{\alpha\}$	$\{\mathfrak{r}^*\}$	$\{*, \mathfrak{r}\}$	$\{\mathfrak{r}\}$	$\{*, -\mathfrak{r}\}$
e	1	1	1	σ_y