Non-Commutative Topology and Topological Quantisation

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I Models for decorations of lattices

We consider the case in which $\mathcal{L} = \Lambda$ is a regular lattice, but $\mathcal{L}^\bullet : \mathcal{L} \to \mathcal{C}$ is not necessarily periodic. Chosing generators $\{\lambda_1, \cdots, \lambda_d\}$ for Λ we have an action of $\mathbb{Z}^d$: $\alpha_{e_i}(\lambda) = \lambda + \lambda_i$.

- Hilbert space is $\ell^2(\mathcal{L}) \otimes \mathbb{C}^N$.
- H_{xy} is a $N \times N$ matrix. Setting $H_n(x) := H_{x,\alpha_n(x)}$ for $n \in \mathbb{Z}^d$, the Schrödinger equation reads

$$H\Psi(x) = \sum_{y \in \mathcal{L}} H_{xy}\Psi(y) = \sum_{n \in \mathbb{Z}^d} H_n(x)\Psi(\alpha_n(x))$$

Thus

$$H = \sum_{n \in \mathbb{Z}^d} H_n T^n$$

T^n the translation operator on $\ell^2(\mathcal{L})$, $T^n\Psi(x) = \Psi(\alpha_n(x))$

- For each n, $x \mapsto H_n(x)$ is pattern equivariant for $\mathcal{L}^\bullet$.
- $\mathbb{Z}^d \ni n \mapsto \sup_x \|H_n(x)\|$ decays with n (the sum converges).

I Models for decorations of lattices

with external magnetic field

Recall $H = \sum_{n\in\mathbb{Z}^d} H_n T^n$.

- We may add an external magnetic field by means of the Peierl's substitution. This means to replace T^{e_i} by $T_\phi^{e_i}$ where

$$(T_\phi^{e_i}\Psi)(x) = e^{i\phi_{x\,x+\lambda_i}}\Psi(\alpha_{e_i}(x))$$

  the phases being determined by a choice of vector potential for the magnetic field.
- $T_\phi^{e_i}$ and $T_\phi^{e_j}$ no longer commute!

$$T_\phi^{\lambda_1+\lambda_2}\Psi(x) = e^{i(\phi_{x\,x+\lambda_1}+\phi_{x+\lambda_1,x_1+\lambda_2}-\phi_{x+\lambda_1+\lambda_2,x})}T_\phi^{\lambda_1}T_\phi^{\lambda_2}\Psi(x)$$

- The phase factor defines a map $\sigma : \mathbb{Z}^d \times \mathbb{Z}^d \to C(\mathcal{L})$,

$$\sigma(n,m)(x) = e^{i(\phi_{x\,\alpha_n(x)}+\phi_{\alpha_n(x)\,\alpha_{n+m}(x)}-\phi_{\alpha_{n+m}(x),x})}$$

  which satisfies a group 2-cocycle identity.
- If the flux of the magnetic field through the unit cell of $\mathcal{L}$ is irrational (in appropriate units) then, even with a periodic $\mathcal{L}^\bullet$ the Hamiltonian will be aperiodic.

Example: Hofstadter model

- $\mathcal{L} = \{n_1 e_1 + n_2 e_2 | n_i \in \mathbb{Z}\}$ integer lattice in $\mathbb{R}^2$ (ONB).
- Hilbert space $\ell^2(\mathcal{L})$.
- The discrete Laplacian on $\mathcal{L}$ is given by
  $H_{xy} = 1$ if x, y are nearest neighbours, i.e. $y - x = \pm e_i$
  $H_{xy} = 0$ otherwise
- Consider an external constant magnetic field whose flux per unit cell is ϕ. For any choice of vector potential A, the phases satisfy

$$\exp i(\phi_{x\,x+e_1} + \phi_{x+e_1\,x+e_1+e_2} + \phi_{x+e_1+e_2\,x+e_2} + \phi_{x+e_2\,x}) = \exp i\phi$$

- The Hofstadter model is the discrete Laplacian coupled to this magnetic field.

$$H_{xy} = \begin{cases} \exp i\phi_{xy} & \text{if } y - x = \pm e_i \\ 0 & \text{otherwise} \end{cases}$$

H is aperiodic iff $\frac{\phi}{2\pi}$ is irrational.

Any FLC Delone set can be seen as colored distorted lattice.

- In a regular lattice Λ we can choose generators $\lambda_1, \cdots, \lambda_d$ and define the next neighbor in direction i of $\lambda \in \Lambda$ to be $\lambda(i) = \lambda + \lambda_i$.
- This defines a free transitive action α of $\mathbb{Z}^d$ on Λ: $\alpha_{e_i}(\lambda) = \lambda + \lambda_i$.

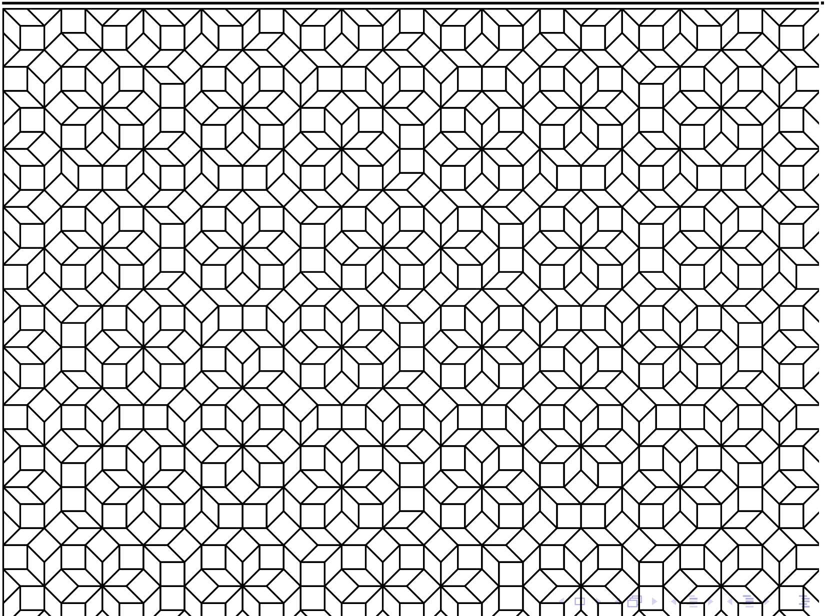
Thm [Sadun-Williams] Given an FLC Delone set $\mathcal{L}'$ one can find a Delone subset $\mathcal{L} \subset \mathcal{L}'$ and define, for each $x \in \mathcal{L}$ and $1 \le i \le d$ the next neighbour $x(i)$ in direction i, in such a way that

1. $\mathcal{L}$ is locally derivable from $\mathcal{L}'$,
2. the choice of $x(i)$ is locally derivable from $\mathcal{L}'$, i.e. depends only on the R-patch of $\mathcal{L}'$ around x up to translation
3. $\alpha_{e_i}(x) = x(i)$ defines a free transitive action of $\mathbb{Z}^d$ on $\mathcal{L}$.

- Fixing a point $x_0 \in \mathcal{L}$ we obtain a bijection between $\mathcal{L}$ and $\mathbb{Z}^d$.
- Encode patterns of $\mathcal{L}'$ around the points of $\mathcal{L}$ by colors: $\mathcal{L}' \cong \mathcal{L}^\bullet$
- Same modelling as for decorations of lattices. We simply say $\mathcal{L}'$ is a decoration of $\mathbb{Z}^d$.

Example: Octagonal tiling as decoration of $\mathbb{Z}^2$

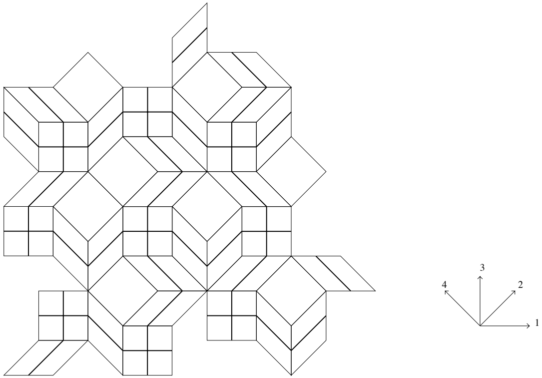
$\mathcal{L}$ = vertices of octagonal tiling



Example: Octagonal tiling as decoration of $\mathbb{Z}^2$

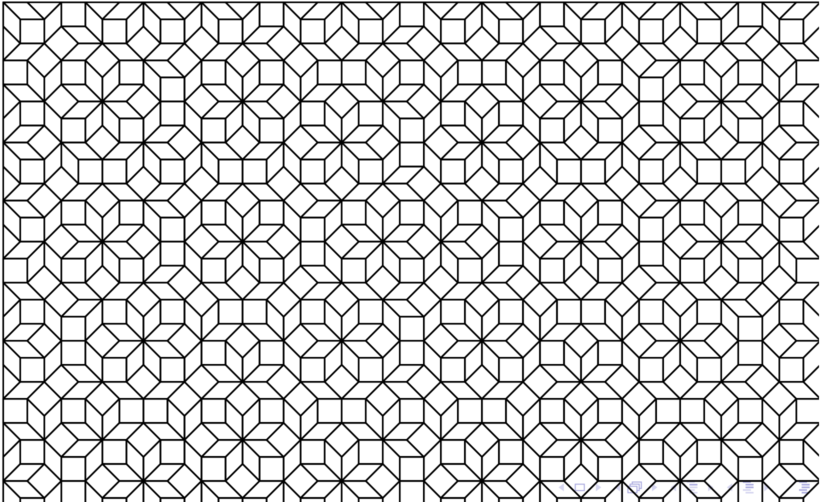
$\mathcal{L}' =$ vertices of octagonal tiling.

- Choose directions 1 and 3
- link opposite edges of direction 1 by a line
- link opposite edges of direction 3 by a line
- line crossings define $\mathcal{L}$
- following wiggly line defines $\mathbb{Z}^2$-action



Example: Discrete Laplacian on octagonal tiling

$\mathcal{L}$ = vertices of octagonal tiling.
Hilbert space $\ell^2(\mathcal{L})$ (no internal degrees of freedom)
$H_{xy} = 1$ if x and y are linked by an edge
$H_{xy} = 0$ otherwise



I Models for insulators

Models with boundary (half space models) 1

To study boundary effects we consider models with a boundary.

- Suppose that bulk model is described by a decoration of a regular lattice $\mathcal{L}^\bullet : \mathcal{L} \to \mathcal{C}$

$$\mathcal{L} = \{\sum_i^d n_i \lambda_i \,|\, n_i \in \mathbb{Z}\}$$

- Let $\hat{\mathcal{L}}$ be the restriction of $\mathcal{L}$ to the halfspace $\mathbb{R}^{d-1} \times \mathbb{R}^+$

$$\hat{\mathcal{L}} = \{\sum_i^d n_i \lambda_i \,|\, n_1, \cdots n_{d-1} \in \mathbb{Z}, n_d \in \mathbb{N}\}.$$

- Let $\hat{\mathcal{L}}^\bullet$ be the restriction of $\mathcal{L}^\bullet$ to $\hat{\mathcal{L}}$.

I Models for insulators

Models with boundary (half space models) 2

Recall $\hat{\mathcal{L}} = \{\sum_i^d n_i \lambda_i \,|\, n_1, \cdots n_{d-1} \in \mathbb{Z}, n_d \in \mathbb{N}\}$.

- Let $\Pi \in \mathcal{B}(\ell^2(\mathcal{L}) \otimes \mathbb{C}^N)$ the orthogonal projection onto $\ell^2(\hat{\mathcal{L}}) \otimes \mathbb{C}^N$.
- Hamiltonian $\hat{H}$ of model with boundary is the compression of the Hamiltonian H of the bulk model to $\hat{\mathcal{L}}$

$$\hat{H} := \Pi H \Pi$$

- This is like taking Dirichlet boundary conditions at the boundary.

II C^*-algebras for insulators

C^*-algebras as background space

Recall: To define continuous deformation of insulators (self-adjoint invertible operators) we need a background topological space.

- Notion of topological phase depends on choice of background space. Different choices = different notions of topological phase.
- $\mathcal{B}(\mathcal{H})$ as background space is not good
  - with norm topology there would be only one phase
  - strong or weak topology not suited
- Smallest norm-closed subalgebra containing all finite range Hamiltonians on $\ell^2(\mathcal{L})$?
- I prefere the smallest norm-closed subalgebra containing all finite range Hamiltonians on $\ell^2(\mathcal{L})$ which satisfy our locality requirement

  H_{xy} depends only on local environment of x, y in $\mathcal{L}^\bullet$
- There is a functorial construction of that algebra, as a twisted crossed product.

$$A_{\mathcal{L}^\bullet,\phi} := C_{\mathcal{L}^\bullet}(\mathcal{L}) \rtimes_{\alpha,\phi} \mathbb{Z}^d \otimes M_N(\mathbb{C}).$$

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$$A_{\mathcal{L}^\bullet,\phi} := C_{\mathcal{L}^\bullet}(\mathcal{L}) \rtimes_{\alpha,\phi} \mathbb{Z}^d \otimes M_N(\mathbb{C}).$$

The crossed product $B \rtimes_\alpha \mathbb{Z}^d$

Let B a C^*-algebra with $\mathbb{Z}^d$-action α.

$B_\alpha\mathbb{Z}$ is space of finite sums of $\sum_{n\in\mathbb{Z}^D} b_n[n]$, $b_n \in B$ with relations

$$b_n[n]b_m[m] = b_n\alpha_n(b_m)[n+m], \quad (b[n])^* = [-n]b^*$$

$B \rtimes_\alpha \mathbb{Z}^d$ is the universal C^*-closure of $B_\alpha\mathbb{Z}$.
closure in $\|a\| = \sup_\pi \|\pi(a)\|_{\mathcal{H}_\pi}$, π $*$-representation on $\mathcal{H}_\pi$.
If there is a twisting cocycle σ change product similar to Peierl's substitution

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Apply this to our case: $B = C_{\mathcal{L}^\bullet}(\mathcal{L})$ and $\alpha(f)(x) = f(\alpha(x))$. The representation

$$\pi_{\mathcal{L}^\bullet}(f)\psi(x) = f(x)\psi(x), \quad \pi_{\mathcal{L}^\bullet}([n]) = T_\phi^n$$

is faithful.

What is $A_{\mathcal{L}^\bullet,\phi}$?

$A_{\mathcal{L}^\bullet,\phi}$ is the smallest norm-closed subalgebra containing all finite range Hamiltonians on $\ell^2(\mathcal{L})$ which satisfy our locality requirement. It is a faithful representation of

$$C_{\mathcal{L}^\bullet}(\mathcal{L}) \rtimes_{\alpha,\phi} \mathbb{Z}^d \otimes M_N(\mathbb{C}).$$

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1. Suppose $\mathcal{L} = \Lambda$ is lattice and $|\mathcal{C}| = 1$, i.e. $\mathcal{L}^\bullet = \mathcal{L}$, $N = 1$
   - $C_{\mathcal{L}^\bullet}(\mathcal{L}) = \mathbb{C}$ and $\alpha = \mathrm{id}$
   - ($d = 2$) $A_{\mathcal{L}^\bullet,\phi} = \mathbb{C} \rtimes_{\mathrm{id},\phi} \mathbb{Z}^2 \cong \mathbb{T}^2_\phi$ non-commutative torus. This algebra underlies the Hofstadter model.
2. Suppose $\mathcal{L}^\bullet$ is Λ-periodic, no external magnetic field, $N = 1$.
   - $C_{\mathcal{L}^\bullet}(\mathcal{L}) = \mathbb{C}^K$ where $K = |\mathcal{L}/\Lambda|$.
   - $A_{\mathcal{L}^\bullet,\phi} = \mathbb{C}^K \rtimes_\alpha \mathbb{Z}^d \cong M_K(\mathbb{T}^d)$ This algebra underlies the Haldane model ($d = 2$, $K = 2$).
3. Suppose $\mathcal{L}^\bullet$ is an aperiodic decoration of a lattice $\mathcal{L} = \Lambda$, no external magnetic field, $N = 1$.
   - $C_{\mathcal{L}^\bullet}(\mathcal{L}) \cong C(\Xi)$, Ξ called the hull of $\mathcal{L}^\bullet$. This algebra underlies the Kohomoto model ($d = 1$).

The hull of $\mathcal{L}^\bullet$

$\mathcal{L}^\bullet : \mathcal{L} \to \mathcal{C}$ is a decoration of a lattice $\mathcal{L} = \Lambda$

$\mathcal{L}^\bullet \in \mathcal{C}^{\mathcal{L}}$ the infinite Cartesian product (color is coordinate at $x \in \mathcal{L}$).

- Product topology on $\mathcal{C}^{\mathcal{L}}$ is compact and totally disconnected
  - $U_R(\xi) := \{\xi' : \mathcal{L} \to \mathcal{C} \mid B_R[\xi] = B_R[\xi']\}$, $R > 0$, $\xi \in \mathcal{C}^{\mathcal{L}}$ is a base of topology. It is closed!
- $\Xi := \overline{\{\mathcal{L}^\bullet - x \mid x \in \mathcal{L}\}}$ hull of $\mathcal{L}^\bullet$.
  - Ξ is equal to the set of all colorings $\mathcal{L} \to \mathcal{C}$ whose R-patches occur somewhere in $\mathcal{L}^\bullet$.

Thm. The map $C(\Xi) \ni \tilde{f} \mapsto f \in C_{\mathcal{L}^\bullet}(\mathcal{L}),\ f(x) = \tilde{f}(\mathcal{L}^\bullet - x)$ is an isomorphism of algebras.

For Kohomoto model on $\ell^2(\mathbb{Z})$ with potential

$$H_{xx} = \begin{cases} 0 & \text{if fractional part of } x\theta \geq 1 - \theta \\ 1 & \text{otherwise} \end{cases}$$

Ξ is $S^1 = \mathbb{R}/\mathbb{Z}$ cut up along $\theta\mathbb{Z}$. (cut up noncommutative torus)

C^*-algebra associated to model with boundary

Recall orthogonal projection $\Pi : \ell^2(\mathcal{L}^\bullet) \otimes \ell^2(\hat{\mathcal{L}}^\bullet)$.

- Most natural: define the algebra for model with boundary as $\Pi \pi_{\mathcal{L}^\bullet}(A_{\mathcal{L}^\bullet,\phi})\Pi$.
  - Problem: we want a functorial definition of this algebra, the above may turn out to small for our purposes.
- The functorial definition of this algebra $\hat{A}_{\mathcal{L}^\bullet,\phi}$ is as the Toeplitz algebra of a crossed product algebra with $\mathbb{Z}$.
  - We take a gauge for the external magnetic field which allows to rewrite the twisted crossed product $C_{\mathcal{L}^\bullet}(\mathcal{L}) \rtimes_{\alpha,\phi} \mathbb{Z}^d$ as an iterated crossed product $A_{\mathcal{L}^\bullet,\phi} = B \rtimes_{\alpha^\perp} \mathbb{Z}$ with $B = C_{\mathcal{L}^\bullet}(\mathcal{L}) \rtimes_{\alpha^\parallel,\phi} \mathbb{Z}^{d-1}$.In that gauge the Peierl's substitution does not alter the translation operators perpendicular to the boundary.
- $\hat{A}_{\mathcal{L}^\bullet,\phi}$ can be faithfully represented as $\Pi \pi_{\mathcal{L}^\bullet}(A_{\mathcal{L}^\bullet,\phi})\Pi$ provided: any R-patch of $\mathcal{L}^\bullet$ appears also in $\hat{\mathcal{L}}^\bullet$.

The Toeplitz-extension $\mathcal{T}(B, \alpha)$

Let B a C^*-algebra with $\mathbb{Z}$-action α.

- $B_\alpha\mathbb{N}$ is finite products of finite sums of $b[n]$, p, $n \in \mathbb{N}$, $b \in B$ with relations
$$[1]b = \alpha(b)[1], \quad b[n][m] = b[n+m]$$
$$[1]^*[1] = [0], \quad [1][1]^* = [0] - p, \quad pb = bp$$
If B is unital we also require that $1[0] = 1$.
- $\mathcal{T}(B, \alpha)$ is the universal $C*$-closure of $B_\alpha\mathbb{N}$.
- Apply this to our case: $B = B_{\mathcal{L}^\bullet}(\mathcal{L})$ and $\alpha = \alpha^\perp$.
- The representation
$$\hat{\pi}_{\mathcal{L}^\bullet}(f[n]) = \Pi\pi_{\mathcal{L}^\bullet}\Pi(f[n])$$
is faithful if, for all f, the restriction of f to $\hat{\mathcal{L}}$ is injective.
The latter is equivalent to $\hat{\mathcal{L}}^\bullet$ having the same R-patches as $\mathcal{L}^\bullet$.

Symmetry protection

Let A be the C^*-algebra for the insulator, $H \in A$ its Hamiltonian.

- An ordinary symmetry is a $\mathbb{C}$-linear automorphism α on A with $\alpha(H) = H$.
- We have three types of "extra ordinary" symmetries $\alpha \in \mathrm{Aut}_{\mathbb{R}}(A)$

  real: anti-linear with $\alpha(H) = H$ (i.e. time reversal)
  cplx anti: $\mathbb{C}$-linear with $\alpha(H) = -H$ (i.e. chiral)
  real anti: anti-linear with $\alpha(H) = -H$ (i.e. particle hole exchange)
- The square of an "extra ordinary" symmetry and the group commutator of "extra ordinary" symmetries is an ordinary symmetry.
- Assuming that the group of symmetries splits we can treat the subgroup G of ordinary symmetry apart by replacing A with A^G.
- We are left with none, one or three "extra ordinary" symmetries of order 2.

Gradings on C^*-algebras

Def. A grading γ on A is a $\mathbb{C}$-linear automorphism of order 2.

- A splits in even and odd part $A = A^+ + A^-$, $A^+ = A^\gamma$.
- A grading is inner if $\gamma = \mathrm{Ad}_\Gamma$ for a self-adjoint unitary (the grading operator) $\Gamma \in A$.
- A grading is balanced if A^- contains a a self-adjoint unitary e. Then $A^- = eA^+$
- The usual extension of γ to $M_n(A)$ is entrywise, notation γ_n
- The standard even extension of γ to $M_2(A)$ is $\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \gamma(a) & -\gamma(b) \\ -\gamma(c) & \gamma(d) \end{pmatrix}$, notation γ_{ev}.

Example: Complex Clifford algebras $\mathbb{C}l_k$

Real structures on C^*-algebras

Def. A real structure $\mathfrak{r}$ on A is an anti-linear automorphism of order 2.
$\mathfrak{r}(ia) = -i\mathfrak{r}(a)$

- A splits in real and imaginary part $A = A^{\mathfrak{r}} + iA^{\mathfrak{r}}$, $A^{\mathfrak{r}}$ is a real C^*-algebra.
- The usual extension of $\mathfrak{r}$ to $M_n(A)$ is entrywise, notation $\mathfrak{r}_n$
  $M_n(A)^{\mathfrak{r}_n} = M_n(A^{\mathfrak{r}})$
- Another important extension of $\mathfrak{r}$ to $M_2(A)$ is

$$\mathfrak{r}^{\mathfrak{h}} = \mathrm{Ad}_{\sigma_y} \circ \mathfrak{r}_2$$

  $M_n(A)^{\mathfrak{r}^{\mathfrak{h}}} = \mathbb{H} \otimes A^{\mathfrak{r}}$ where $\mathbb{H}$ are the quaternions.

Example: Real structures $\mathfrak{l}_{r,s}$ on Clifford algebras and $Cl_{r,s}$

Graded real structures on C^*-algebras

Def. A graded real structure $(\gamma, \mathfrak{r})$ on A is a grading and a real structure which commute.

- A graded real structure is real (or imaginary) inner if the grading operator Γ is real (or imaginary).
- A graded real structure is balanced if A^- contains a real self-adjoint unitary.

Theorem If $(A, \gamma, \mathfrak{r})$ is real inner balanced then

$$(M_2(A), \gamma_2, \mathfrak{r}_2) \cong (A \hat{\otimes} \mathbb{C}l_2, \mathrm{id} \otimes \mathrm{st}, \mathfrak{r} \otimes \mathfrak{l}_{1,1})$$

If $(A, \gamma, \mathfrak{r})$ is imaginary inner balanced then

$$(M_2(A), \gamma_2, \mathfrak{r}_2) \cong (A \hat{\otimes} \mathbb{C}l_2, \mathrm{id} \otimes \mathrm{st}, \mathfrak{r} \otimes \mathfrak{l}_{2,0})$$

The right hand side does not depend on γ.

Topological phases

For $B \subset A$ and $\alpha : A \to A$ denote $B^\alpha = \{b \in B \,|\, \alpha(b) = b\}$.
A topological phase is a connected component of $\mathcal{GL}(A)^{\{\alpha\}}$ where the collection of maps $\{\alpha\}$ expresses the constreints on the Hamiltonian. If there are no summetries, then we have only $\alpha = *$.
The topological phase of H is its homotopy class $[h]$.

symmetry	background space	K-group
(cplx) none	$H \in \mathcal{GL}(A)^{\{*\}}$	$KU_0(A)$
cplx anti	$H \in \mathcal{GL}(A)^{\{*,-\gamma\}}$	$K_1(A,\gamma)$
real	$H \in \mathcal{GL}(A)^{\{*,\mathfrak{r}\}}$	$KO_0(A^\mathfrak{r})$
real anti	$H \in \mathcal{GL}(A)^{\{*,-\mathfrak{r}\}}$	$KO_2(A^\mathfrak{r})$
real & cplx anti	$H \in \mathcal{GL}(A)^{\{*,\mathfrak{r},-\gamma\}}$	$K_1(A^\mathfrak{r},\gamma)$

- 2 purely complex cases
- 2 purely real cases
- 2 graded cases

Topological phases

Balanced inner gradings

Consider 2 graded cases under assumption that $\gamma = \mathrm{Ad}_\Gamma$ (inner).

- Let $\Pi_+ = \frac{1+\Gamma}{2}$ and $A_{++} = \Pi_+ A \Pi_+$. Fix $e \in A^-$, $e^* = e = e^{-1}$.
- We have a bijection $Q_e : \mathcal{GL}(A)^{\{*\}} \to \mathcal{GL}(A_{++})$,

$$Q_e(h) = \Pi_+ e h \Pi_+$$

  Its inverse is $h = eQ_e(h) + Q_e(h)^* e$.
- Topological phases are thus in bijection with connected components of $\mathcal{GL}(A_{++})$

symmetry	background space	K-group
inner cplx anti	$Q_e(H) \in \mathcal{GL}(A_{++})$	$KU_1(A)$
real & real inner c.a.	$Q_e(H) \in \mathcal{GL}(A_{++})^{\{\mathfrak{r}\}}$	$KU_1(A^{\mathfrak{r}})$
real & im. inner c.a.	$Q_e(H) \in \mathcal{GL}(A_{++})^{\{\mathfrak{r}'\}}$	$KU_{-1}(A^{\mathfrak{r}})$

$\mathfrak{r}'(a) = e\mathfrak{r}(a)^* e$ (a transposition)

Example: Haldane model 1

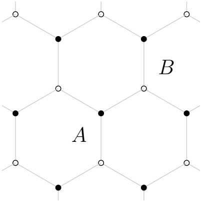
Haldane [Nobel prize 2016] model.

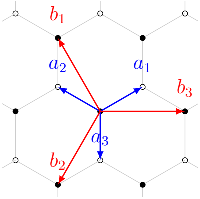
- Let Λ is lattice generated by b_1, b_2 and $b_3 = -b_1 - b_2$.
- $\mathcal{L} = \mathcal{L}_A \cup \mathcal{L}_B$, two orbits, $\mathcal{L}_A = \Lambda$, $\mathcal{L}_B = \Lambda + a_1$ (hexagonal "lattice"), So $\mathcal{L} = \Lambda^\bullet$ with two colors (A- and B-sublattice).
- Real parameters, M, t, t_2, φ.

$$H_{xx} = \begin{cases} M & \text{if } x \in \mathcal{L}_A \\ -M & \text{if } x \in \mathcal{L}_B \end{cases} \qquad \text{onsite, distinguishes sublattices}$$

$$H_{x,x\pm a_i} = t \qquad \text{nearest neighbour}$$

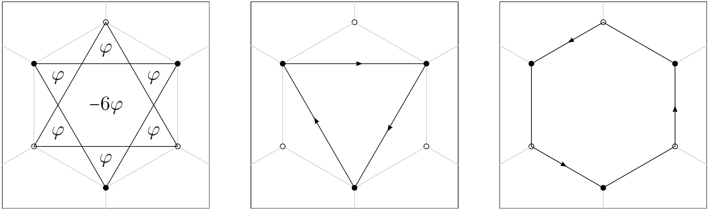
$$H_{x\,x+b_i} = \begin{cases} t_2 e^{i\varphi} & \text{if } x \in \mathcal{L}_A \\ t_2 e^{-i\varphi} & \text{if } x \in \mathcal{L}_B \end{cases}, \; H_{x+b_i\,x} = \overline{H_{x\,x+b_i}} \qquad \text{next n. neighb.}$$





Example: Haldane model 2

- H_{xy} being complex means that there is a magnetic field.
- Flux through the triangle bounded by hoping vectors b_i between next nearest neighbours is 3φ
- Flux through an elementary cell is 0 (magnetic field is internal)



Example: Haldane model 3

- Recall $\mathcal{L} = \Lambda^\bullet$. Rewrite H as a function $H : \Lambda \to M_2(\mathbb{C})$.
- Apply Fourier transformation to $H(\lambda)$. Using Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_0 = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as base of $M_2(\mathbb{C})$ one obtains $\hat{H}(k) = \sum_{i=0}^{3} d_i(k)\sigma_i$, $k \in \mathbb{R}^2/\Lambda^{rec}$.

$$
\begin{aligned}
d_0(k) &= 2t_2 \cos\varphi \sum_{i=1}^{3} \cos(k \cdot b_i) \\
d_1(k) &= t\big(\cos(k \cdot b_1) + \cos(k \cdot b_2) + 1\big) \\
d_2(k) &= t\big(\sin(k \cdot b_1) - \sin(k \cdot b_2)\big) \\
d_3(k) &= M - 2t_2 \sin\varphi \sum_{i=1}^{3} \sin(k \cdot b_i)
\end{aligned}
$$

Example: Haldane model 5

Spektrum and band touching

Since $(\hat{H}(k) - d_0(k)\mathbb{1})^2 = \sum_{i=1}^{3} d_i(k)^2 \mathbb{1}$ we have

$$E(k) = \pm\sqrt{\sum_{i=1}^{3} d_i(k)^2} - d_0(k)$$

so band touching if for all $i = 1, 2, 3$ $d_i(k) = 0$.
$d_2(k) = 0$ and $d_1(k) = 0$ yield together the constreint

$$k \cdot b_1 - k \cdot b_2 \in 2\pi\mathbb{Z}, \quad \cos(k \cdot b_1) = -\frac{1}{2}$$

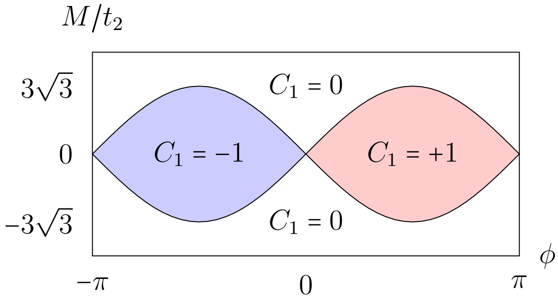
This has two solutions, expressed in reciprocal base (mod Λ^{rec})

$$k = \pm\frac{1}{3}(b_1^* + b_2^*)$$

With this solution we have $d_3(k) = M - t_2 3\sqrt{3}\sin(\phi)$ and so band touching condition is

$$M = t_2 3\sqrt{3}\sin(\phi)$$

Example: Haldane model 6



[Fruchart & Carpentier 2013] An Introduction to Topological Insulators, *arXiv* cond-mat.mes-hall 1310.0255