Introduction to semigroup C^*-algebras, and KMS states for right LCM semigroup C^*-algebras

Nathan Brownlowe
(With Zahra Afsar, Nadia S. Larsen, and Nicolai Stammeier)

School of Mathematics and Statistics
The University of Sydney

21 July 2017

C^*-algebras associated to groups

There are two natural C^*-algebras we associate to a discrete group G.

- For each $g \in G$ the operator $\lambda_g \in B(\ell^2(G))$ characterised by

$$\lambda_g \delta_h = \delta_{gh}$$

  is a unitary operator with adjoint $\lambda_g^* = \lambda_{g^{-1}}$. The **reduced group C^*-algebra of G** is the C^*-algebra $C_r^*(G)$ generated by $\{\lambda_g : g \in G\} \subseteq B(\ell^2(G))$.
- The other C^*-algebra we associate to G is the **group C^*-algebra** $C^*(G)$ which comes with a unitary representation $i_G : G \to C^*(G)$, universal in the following sense: if $u : G \to U(A)$ is a unitary representation of G into the group of unitaries in a C^*-algebra A, then there is a homomorphism $C^*(G) \to A$ satisfying $i_G(g) \mapsto u_g$ for all $g \in G$.

Fact:
$C^*(G) \cong C_r^*(G) \iff G$ is amenable $\iff C^*(G)$ is nuclear.

Semigroup C^*-algebras

Suppose S is a semigroup with identity, and which is left cancellative, in the sense that $rs = rt \implies s = t$ for all $r, s, t \in S$.

How do we associate C^*-algebras to S?

For each $s \in S$ consider the operator $V_s \in B(\ell^2(S))$ characterised by $V_s \delta_t = \delta_{st}$. The adjoint of V_s is characterised by

$$V_s^* \delta_t = \begin{cases} \delta_{s'} & \text{if } t = ss' \\ 0 & \text{otherwise} \end{cases}$$

So $V_s^* V_s = 1$, which means V_s is an **isometry**.

The **reduced semigroup C^*-algebra of S** (or the **Toeplitz algebra of S**) is the C^*-algebra $C_r^*(S)$ generated by $\{V_s : s \in S\} \subseteq B(\ell^2(S))$.

A universal semigroup C^*-algebra?

What about a universal C^*-algebra for semigroups?

The first guess is to follow the group-case lead and associate to each S a C^*-algebra which is universal for isometric representations of S.

The good news? This construction works for $\mathbb{N}$:

Theorem (Coburn '67)

All non-unitary isometries generate isomorphic C^*-algebras.

A universal semigroup C^*-algebra?

The bad news is that it doesn't take long for this universal construction to break down.

Murphy [11] studied such a universal C^*-algebra associated to $\mathbb{N}^2$, and showed that this C^*-algebra is not nuclear. This led Murphy to say $\mathbb{N}^2$ is not amenable, which would be an unsatisfactory part of any amenability theory for semigroups.

By only considering isometric representations of $\mathbb{N}^2$ we are losing too much information: the isometries $V_{(1,0)}, V_{(0,1)} \in B(\ell^2(\mathbb{N}^2))$ satisfy

$$V^*_{(1,0)} V_{(0,1)} = V_{(0,1)} V^*_{(1,0)} \quad \text{and} \quad V^*_{(0,1)} V_{(1,0)} = V_{(1,0)} V^*_{(0,1)}.$$

Should this behaviour be factored into the construction of any universal C^*-algebras?

Right LCM semigroups

Nica [13] made the key observation that a universal C^*-algebra of a semigroup should model the ideal structure of the semigroup. He introduced a class of semigroups called quasi-lattice ordered semigroups, and constructed a universal C^*-algebra to each such semigroup.

These semigroups are generalised by right LCM semigroups: a discrete left-cancellative semigroup S is called **right LCM** if the intersection of two principal right ideals is either empty or another principal right ideal. So for each $s, t \in S$ we have

$$sS \cap tS = \begin{cases} rS & \text{if } sS \cap tS \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

Right LCM semigroup C^*-algebras

We can use Li's theory [10] of semigroup C^*-algebras to associate C^*-algebras to right LCM semigroups.

The **full C^*-algebra** $C^*(S)$ of a right LCM semigroup S is

$$C^*\left(v_s \;\middle|\; v_s^* v_s = 1,\, v_s v_t = v_{st},\, v_s v_s^* v_t v_t^* = \begin{cases} v_r v_r^* & \text{if } sS \cap tS = rS \\ 0 & \text{if } sS \cap tS = \varnothing \end{cases}\right),$$

universal in the sense that for every family of isometries w_s, $s \in S$, in a C^*-algebra B satisfying the above relations, there is a homomorphism $\pi_w : C^*(S) \to B$ satisfying $\pi(v_s) = w_s$.

Right LCM semigroup C^*-algebras

The relation

$$v_s v_s^* v_t v_t^* = \begin{cases} v_r v_r^* & \text{if } sS \cap tS = rS \\ 0 & \text{if } sS \cap tS = \varnothing \end{cases}$$

is equivalent to

$$v_s^* v_t = \begin{cases} v_{s'} v_{t'}^* & \text{if } sS \cap tS = ss'S,\ ss' = tt' \\ 0 & \text{if } sS \cap tS = \varnothing, \end{cases}$$

and it follows that

$$C^*(S) = \overline{\text{span}}\{v_s v_t^* : s, t \in S\}.$$

Examples of right LCM semigroups

Examples of right LCM semigroups include

- Groups
- Free monoids $\mathbb{F}_n^+$ and free abelian monoids $\mathbb{N}^k$
- Right-angled Artin monoids: for $\Gamma = (V, E)$ a graph

$$A_\Gamma = \langle v \in V \mid vw = wv \text{ if } (v, w) \in E \rangle.$$

- Braid monoids and Garside monoids
- Zappa-Szép products $X^* \bowtie G$ associated to self-similar actions $G \curvearrowright X^*$
- Baumslag–Solitar monoids $BS(c, d)^+ = \langle a, b \mid ab^c = b^d a \rangle^+$, $cd > 1$
- $\mathbb{N} \rtimes \mathbb{N}^\times$, $R \rtimes R^\times$ for a principal ideal domain R
- Semi-direct products $G \rtimes_\theta P$ coming from algebraic dynamical systems (G, P, θ).

Motivation for KMS states

Some work in the literature:

- Laca–Raeburn [6] calculated the KMS structure for the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$.
- Laca–Ramagge–Raeburn [7] calculated the KMS structure for Exel crossed products associated to dilation matrices.
- Laca–Ramagge–Raeburn–Whittaker [8] calculated the KMS structure of the Toeplitz algebra associated to a self-similar action
- Clark–an Huef–Raeburn [3] calculated the KMS structure of the Toeplitz algebra of Baumslag–Solitar semigroups.

These C^*-algebras can all be described as right LCM semigroup C^*-algebras.

Some distinguished substructures

For S a right LCM semigroup we care about a number of substructures:

- The **units**

$$S^* := \{\text{invertible elements in } S\}$$

- The **core** (Crisp–Laca [4], Starling [15])

$$S_c := \{a \in S : aS \cap sS \neq \varnothing \text{ for all } s \in S\}$$

- The **core irreducibles** (Stammeier [14])

$$S_{ci} := \{s \in S : s = ta,\ a \in S_c \implies a \in S^*\}$$

Example: For $S = \mathbb{N} \rtimes \mathbb{N}^\times$, we have $S^* = \{(0,1)\}$, $S_c = \mathbb{N} \times \{1\}$, and

$$S_{ci} = \{(m,a) : a \in \mathbb{N}^\times,\ 0 \leq m < a\}.$$

Self-similar actions

Let X be a finite alphabet, and X^* the set of finite words. Under concatenation, X^* is a semigroup with identity the empty word $\varnothing$. A faithful action of a group G on X^* is **self-similar** if for every $g \in G$ and $x \in X$, there exists unique $g|_x \in G$ such that

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w).$$

(Grigorchuk [5], Nekrasheyvech [12].)

The **Zappa-Szép product** $X^* \bowtie G$ is the set $X^* \times G$ with product

$$(w, g)(z, h) = (w(g \cdot z), g|_z h).$$

(Brin [1], Lawson [9].)

For $S = X^* \bowtie G$ we have

$$S_c = S^* = \{\varnothing\} \times G \quad \text{and} \quad S_{ci} = X^* \times G.$$

Some more structure in the semigroups

We can use the core S_c to define an equivalence relation:

$$s \sim t \iff sa = tb \quad \text{for some } a, b \in S_c,$$

and an action $\alpha\colon S_c \curvearrowright S/_\sim$ given by $\alpha_a([s]) = [as]$.

A finite subset F of S is a **foundation set** if for every $s \in S$ there is $f \in F$ such that $sS \cap fS \neq \varnothing$.

A foundation set $F \subset S$ is **accurate** if $fS \cap f'S = \varnothing$ for all $f, f \in F$ with $f \neq f'$.

admissible semigroups

We can calculate the KMS structure of $C^*(S)$ for the following class of semigroups:

A right LCM semigroup S is called **admissible** if

- $S = S_{ci}^1 S_c$
- any right LCM in S of a pair of elements in S_{ci} belongs to S_{ci}
- there is a nontrivial homomorphism of monoids $N\colon S \to \mathbb{N}^\times$ satisfying
  - $|N^{-1}(n)/_\sim| = n$ for all $n \in N(S)$; and
  - for each $n \in N(S)$, every transversal of $N^{-1}(n)/_\sim$ is an accurate foundation set for S.
  - the monoid $N(S)$ is the free abelian monoid in $\mathrm{Irr}(N(S))$.

The dynamics

The existence of a nontrivial homomorphism $N\colon S \to \mathbb{N}^\times$ gives an action $\sigma : \mathbb{R} \to \mathrm{Aut}\, C^*(S)$, where

$$\sigma_x(v_s) = N_s^{ix} v_s \quad \text{for each } x \in \mathbb{R} \text{ and } s \in S.$$

Note that each $v_s v_t^*$ is σ-analytic.

For $\beta \in \mathbb{R}$ we define

$$\zeta_S(\beta) := \sum_{n \in N(S)} n^{-(\beta-1)}.$$

The **critical inverse temperature** $\beta_c \in \mathbb{R} \cup \{\infty\}$ is the smallest value so that $\zeta_S(\beta) < \infty$ for all $\beta \in \mathbb{R}$ with $\beta > \beta_c$.

Some generalised scales and critical inverse temperatures

1. For $S = \mathbb{N} \rtimes \mathbb{N}^{\times}$, we take $N(m, a) = a$. We have

$$\zeta_S(\beta) = \sum_{n \in N(S)} n^{-(\beta-1)} = \sum_{a=1}^{\infty} a^{-(\beta-1)},$$

which is finite iff $\beta > 2$. Hence $\beta_c = 2$.

2. For $S = X^* \bowtie G$, we take $N(w, g) = |X|^{\ell(w)}$. We have

$$\zeta_S(\beta) = \sum_{n \in N(S)} n^{-(\beta-1)} = \sum_{\ell=0}^{\infty} \left(|X|^{\ell}\right)^{-(\beta-1)} = \sum_{\ell=0}^{\infty} \left(|X|^{-(\beta-1)}\right)^{\ell},$$

which is finite iff $|X|^{-(\beta-1)} < 1$ iff $\beta > 1$. Hence $\beta_c = 1$.

On the action $S_c \curvearrowright S/_\sim$

Recall the action $\alpha\colon S_c \curvearrowright S/_\sim$ given by $\alpha_a([s]) = [as]$.

We say α is **faithful** if

$$a, b \in S_c,\ a \neq b \Longrightarrow \alpha_a([s]) \neq \alpha_b([s]) \text{ for some } [s].$$

We say α is **almost free** if

$$\left|\{[s] : \alpha_a([s]) = \alpha_b([s])\}\right| < \infty \quad \text{for all } a, b \in S_c,\ a \neq b.$$

The KMS Theorem

We denote $\varphi\colon C^*(S_c) \to C^*(S)$ given by $\varphi(w_a) = v_a$ for all $a \in S_c$, and τ the trace $\tau(w_a w_b^*) = \delta_{a,b}$ on $C^*(S_c)$.

Theorem (Afsar–B–Larsen–Stammeier)

Suppose S is an admissible semigroup and σ the action defined above.

- *There are no KMS_β states on $C^*(S)$ for $\beta < 1$.*
- *For $\beta > \beta_c$, there is an affine homeomorphism between KMS_β states on $C^*(S)$ and normalised traces on $C^*(S_c)$.*
- *If α is almost free, then for each $\beta \in [1, \beta_c]$ there is a unique KMS_β state ψ_β determined by $\psi_\beta \circ \varphi = \tau$.*
- *If $\beta_c = 1$, α is faithful, and S has finite propagation, then there are numbers $\kappa_{a,b} \in [0,1]$ for each pair $a, b \in S_c$, a trace ρ on $C^*(S_c)$ given by $\rho(w_a w_b^*) = \kappa_{a,b}$, and a unique KMS_1 state ψ_1 determined by $\psi_1 \circ \varphi = \rho$.*

The quotients

B–Ramagge–Robertson–Whittaker [2]: the **boundary quotient** C^*-algebra $\mathcal{Q}(S)$ is the quotient of $C^*(S)$ by the relation

$$\prod_{f \in F} (1 - v_f v_f^*) = 0 \quad \text{for all foundation sets } F \subset S.$$

Stammeier [14]: The **core boundary quotient** C^*-algebra $\mathcal{Q}_c(S)$ is the quotient of $C^*(S)$ by the relation above, except for all foundation sets $F \subset S_c$. But this amounts to the relation

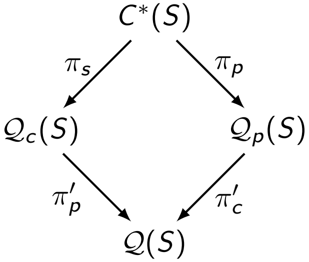
$$v_a v_a^* = 1 \text{ for all } a \in S_c.$$

The **proper boundary quotient** C^*-algebra $\mathcal{Q}_p(S)$ is the quotient of $C^*(S)$ by the relation

$$\prod_{f \in F} (1 - v_f v_f^*) = 0 \quad \text{for all accurate foundation sets } F \subset S_{ci}.$$

The KMS Theorem Part 2

The **boundary quotient diagram** is



Theorem (ABLS)

Suppose S is an admissible semigroup.

- *Every KMS_β-state factors through π_c.*
- *A KMS_β-state factors through π_p if and only if $\beta = 1$.*
- *A KMS_β state factors through $\pi\colon C^*(S) \to \mathcal{Q}(S)$ if and only if $\beta = 1$.*

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Thanks!