INTRODUCTION TO KMS STATES

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Abstract. Here are some notes from a series of lectures at the CIMPA Research School "Noncommutative geometry and applications to quantum physics", July 12–22, Quy Nhon, Vietnam. This is a typed up version of what I presented on the board (for the first 5 lectures– the 6th was a slides talk on KMS states for semigroup C^*-algebras). For a much more detailed account, see the references within. In particular, see [1] for KMS states, and [5] for the work on the KMS states on graph algebras.

1. Motivation and Background on C^*-algebras

The basic idea behind modelling quantum systems using a C^*-algebra is:

(1) The observables in the system (such as momentum, position etc) correspond to self adjoint elements of a C^*-algebra A.
(2) The states of the quantum system correspond to states on A.
(3) When the system is in state ϕ, the expected value of observable a is given by $\phi(a)$.
(4) The time evolution of a quantum system is governed by an action $\alpha\colon \mathbb{R} \to \mathrm{Aut}(A)$, in the sense that if at initial time t_0 the system is in state ϕ, then at time $t_0 + t$, then system is in state $\phi \circ \alpha_t$.

The theory of KMS states gives a mathematical formalism for describing the states ϕ of the system when it is in equilibrium.

Definition 1.1.

(i) An *algebra over* $\mathbb{C}$ is a vector space A over $\mathbb{C}$ with an associative multiplication which is compatible with the vector-space structure:

$$a(b+c) = ab + ac,\ (a+b)c = ac + bc \text{ and } a(zb) = (za)b = z(ab) \text{ for all } a, b, c \in A \text{ and } z \in \mathbb{C}.$$

(ii) A $*$-*algebra* is an algebra A with an involution: a map $a \mapsto a^*$ satisfying

$$(wa + zb)^* = \overline{w}a^* + \overline{z}b^*,\ (ab)^* = b^*a^* \text{ and } (a^*)^* = a \text{ for all } a, b \in A \text{ and } w, z \in \mathbb{C}.$$

(iii) A *Banach algebra* is a Banach space A over $\mathbb{C}$ which is an algebra over $\mathbb{C}$ with multiplication satisfying

$$\|ab\| \le \|a\|\,\|b\| \text{ for all } a, b \in \mathbb{C}.$$

(iv) A C^*-*algebra* is a Banach algebra A over $\mathbb{C}$ with an involution $a \mapsto a^*$ satisfying the C^*-identity:

$$\|a^*a\| = \|a\|^2 \text{ for all } a \in A.$$

Exercise 1.2. *Let H be a Hilbert space, and $B(H)$ the collection of all bounded linear operators on H. Show that $B(H)$ is a C^*-algebra under the usual operations, operator norm $\|T\|_{\mathrm{op}} := \sup\{\|Th\| : h \in H,\ \|h\| \le 1\}$, and the usual adjoint on $B(H)$.*

In fact, all C^*-algebras look like a collection of bounded linear operators on Hilbert space:

Theorem 1.3 (Gelfand–Nairmark)**.** *Every C^*-algebra has a faithful representation on Hilbert space.*

The GNS-construction (after Gelfand, Nairmark and Segal) gives you such a faithful representation.

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Exercise 1.4. *Let X be a locally compact Hausdorff space. We denote by*

$$C_0(X) := \{f\colon X \to \mathbb{C} : f \text{ continuous}, \{x : |f(x)| \geq \epsilon\} \text{ compact for all } \epsilon > 0\}.$$

Show that $C_0(X)$ is a C^*-algebra under pointwise operations, sup-norm $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$, and involution $f^*(x) = \overline{f(x)}$.

Note that $C_0(X)$ is an example of a commutative C^*-algebra, because $(fg)(x) = f(x)g(x) = g(x)f(x) = (gf)(x)$. We will now explain that *all* commutative C^*-algebras look like $C_0(X)$ for some X.

Definition 1.5. (i) The *dual space* of a C^*-algebra A is the set of bounded linear functionals $\phi\colon A \to \mathbb{C}$. We denote the dual space by A^*. The dual space is a Banach space under the operator norm. A sequence of linear functionals (ϕ_n) in A^* converges weak-$*$ to $\phi \in A^*$, if $\phi_n(a) \to \phi(a)$ for all $a \in A$.

(ii) The *maximal ideal space* Δ_A of a commutative Banach algebra A is the set of nonzero homomorphisms from A to $\mathbb{C}$; it is a locally compact Hausdorff space, when given the topology induced from the weak-$*$ topology from A^*. We call this space the maximal ideal space because the map that takes each element of Δ_A to its kernel is a bijection onto the set of maximal ideals of A.

Definition 1.6. The *Gelfand transform* Γ is the map $\Gamma\colon A \to C_0(\Delta_A)$ given by $\Gamma(a)(\phi) = \phi(a)$. We usually write $\hat{a}$ for $\Gamma(a)$.

Theorem 1.7 (Gelfand–Naimark)**.** *If A is a commutative C^*-algebra, then the Gelfand transform $\Gamma\colon A \to C_0(\Delta_A)$ is an isometric $*$-isomorphism.*

Definition 1.8. Let A be a unital Banach algebra. The *spectrum* of $a \in A$ is the set

$$\sigma_A(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\}.$$

An element a in a C^*-algebra A is *self-adjoint* if $a = a^*$.

Theorem 1.9. *Suppose A is a unital C^*-algebra and $a \in A$ is self-adjoint. Then $\sigma(a) \subseteq [0, \infty)$ if and only if $a = b^*b$ for some $b \in A$.*

Definition 1.10. Let A be a C^*-algebra, and $a, b \in A$. We say that $a \geq b$ is $a - b = c^*c$ for some $c \in A$. We say a is *positive* if $a = c^*c$ for some $c \in A$. If H is a Hilbert space and $A = B(H)$, an operator T is positive if and only if $(Th|h) \geq 0$ for all $h \in H$.

Definition 1.11. A bounded linear functional ϕ on A is *positive* if $\phi(a^*a) \geq 0$ for all $a \in A$. (So if it takes positive elements on A to positive elements of $\mathbb{R}$.) A positive functional ϕ if a *state* if $\|\phi\| = 1$. A positive linear functional τ is a *trace* if $\tau(ab) = \tau(ba)$ for all $a, b \in A$, and is a *tracial state* (or a *normalised trace*) if it is both a trace and a state.

Exercise 1.12. *Prove that the following are states.*

(i) *Let $A = C(X)$, the continuous functions on a compact Harusdorff space X, and for each $x \in X$, let ϕ_x be the evaluation map $\phi_x(f) := f(x)$.*

(ii) *Let $A = C(X)$, and for each finite Borel probability measure μ on X, let ϕ_μ be given by*

$$\phi_\mu(f) := \int_X f \, d\mu.$$

(iii) *Let A be a C^*-algebra, and for each representation $\pi\colon A \to B(H)$ and unit vector $h \in H$, let ϕ_h be given by $\phi_h(a) = (\pi(a)h|h)$. (Such states are called* vector states.*)*

Exercise 1.13. *Let H be a Hilbert space, and $\{e_j : j \in J\}$ an orthonormal basis for H. We define the* trace *of a positive operator T to be*

$$\operatorname{Tr}(T) := \sum_{j \in J} (Te_j|e_j) \in [0, \infty].$$

Show that Tr *does not depend on the choice of orthonormal basis by showing*

(i) $\mathrm{Tr}(T^*T) = \mathrm{Tr}(TT^*)$ *for all* $T \in B(H)$*; and*
(ii) $\mathrm{Tr}(UTU^*) = \mathrm{Tr}(T)$ *for all* $T \geq 0$ *and unitaries* $U \in B(H)$.

Show that Tr *is a trace on* $B(H)$.

2. Introduction to KMS states

We work with *inverse temperature* $\beta = 1/k_B T$, where k_B is Boltzmann's constant.

To motivate the KMS condition, consider a finite quantum system. (See [6] or [1] for more details.) Let $A = M_n(\mathbb{C})$. Every time evolution on $M_n(\mathbb{C})$ is given by an action $\alpha\colon \mathbb{R} \to \mathrm{Aut}(M_n(\mathbb{C}))$ of the form

$$\alpha_t(a) = e^{itH} a e^{-itH},$$

for some self-adjoint matrix $H \in M_n(\mathbb{C})$ (the Hamiltonian). We say $Q \in M_n(\mathbb{C})$ is a *density matrix* if $Q \geq 0$ and $\mathrm{Tr}(Q) = 1$. There is a one-to-one correspondence between states ϕ of $M_n(\mathbb{C})$ and density matrices Q_ϕ such that $\phi(a) = \mathrm{Tr}(Q_\phi a)$.

Exercise 2.1. *Show that* $\phi = \phi \circ \alpha_t \iff [Q_\phi, H] = 0$.

The *free energy* of ϕ with Hamiltonian H at inverse temperature β is $F(\phi) := -\mathrm{Tr}(Q_\phi \log Q_\phi) + \beta\phi(H)$. The equilibrium state is the state of minimal free energy, and it's given by the *Gibbs state*

$$\rho_\beta(a) = \frac{\mathrm{Tr}(e^{-\beta H} a)}{\mathrm{Tr}(e^{-\beta H})}.$$

We have the following fact:

Exercise 2.2. *A state* ϕ *on* $M_n(\mathbb{C})$ *satisfies*

$$\phi(ab) = \phi(b\alpha_{i\beta}(a)) \text{ for all } a, b \in M_n(\mathbb{C}) \tag{2.1}$$

if and only if $\phi = \rho_\beta$.

Condition (2.1) is the KMS condition.

We need a number of definitions to set up the general form of the KMS condition.

Definition 2.3. Let A be a C^*-algebra, and Ω an open subset of $\mathbb{C}$. A function $f\colon \Omega \to A$ has a derivative at $z_0 \in \Omega$ if

$$f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists, where the limit is in the norm topology. The function f is *analytic* if f' exists and is continuous on Ω.

Definition 2.4. A *C^*-algebraic dynamical system* is a pair (A, α) consisting of a C^*-algebra A, and a strongly continuous action $\alpha\colon \mathbb{R} \to \mathrm{Aut}(A)$. By strongly continuous we mean that the map $t \mapsto \alpha_t(a)$ is a continuous map from $\mathbb{R}$ into A for each fixed $a \in A$.

From here we fix a C^*-dynamical system (A, α). For each $z \in \mathbb{C}$ we denote by $S(z) := \{w \in \mathbb{C} : \mathrm{Im}(w) \in [0, \mathrm{Im}(z)]\}$, if $\mathrm{Im}(z) \geq 0$, and $S(z) := \{w \in \mathbb{C} : \mathrm{Im}(w) \in [\mathrm{Im}(z), 0]\}$, if $\mathrm{Im}(z) < 0$. We denote by $S(z)^0$ is the interior of $S(z)$.

Definition 2.5. Fix $z \in \mathbb{C}$. We define the map $\alpha_z\colon A \to A$ by first setting its domain, $\mathrm{Dom}(\alpha_z)$, to be the set of all $a \in A$ such that there exists a (necessarily unique) function $f\colon S(z) \to A$ such that

(i) f is continuous;
(ii) f is analytic on $S(z)^0$; and
(iii) $\alpha_t(a) = f(t)$.

We then define $\alpha_z\colon A \to A$ by $\alpha_z(a) = f(z)$.

Definition 2.6. An element $a \in A$ is *α-analytic* if the function $\mathbb{R} \to A$ given by $t \mapsto \alpha_t(a)$ extends to an entire function $\mathbb{C} \to A$ given by $z \mapsto \alpha_z(a)$.

Exercise 2.7. *Show that α-analytic elements are dense, by showing that for each $a \in A$ and each $n \in \mathbb{N}$, the element*

$$a_n := \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_t(a) e^{-nt^2}\, dt$$

is α-analytic, and $a_n \to a$ as $n \to \infty$.

We will now define a KMS state.

Definition 2.8. Let (A, α) be a C^*-dynamical system. For $\beta > 0$, we say a state ϕ is a *KMS$_\beta$ state* if

$$\phi(ab) = \phi(b\alpha_{i\beta}(a))$$

for all α-analytic elements $a, b \in A$. We require that a KMS$_0$ state be α-invariant. A *KMS$_\infty$ state* is a weak-$*$ limit of KMS$_{\beta_n}$ states as $\beta_n \to \infty$. A *ground state* ϕ is such that $z \mapsto \phi(b\sigma_z(a))$ is bounded on the upper-half plane for all α-analytic elements $a, b \in A$.

Theorem 2.9. *Let (A, α) be a C^*-dynamical system, ϕ a state on A, and $0 < \beta < \infty$. The following are equivalent:*

(i) *ϕ is a KMS$_\beta$ state;*
(ii) *$\phi(ab) = \phi(b\alpha_{i\beta}(a))$ for all a and b in a set of α-analytic elements in an α-invariant dense linear span; and*
(iii) *for all $a, b \in A$ there is a bounded continuous function f on the strip*

$$\Omega_\beta = \{z \in \mathbb{C} : 0 \le \mathrm{Im}(z) \le \beta\}$$

such that f is holomorphic in the interior of Ω_β, and

$$f(t) = \phi(b\alpha_t(a)) \text{ and } f(t + i\beta) = \phi(\alpha_t(a)b) \text{ for all } t \in \mathbb{R}.$$

We will sketch some of the details of $(i) \iff (iii)$. For $(i) \implies (iii)$, let a and b be α-analytic elements. Define $f_{a,b}(z) := \phi(b\alpha_z(a))$. Then $f_{a,b}$ is entire because $z \mapsto \alpha_z(a)$ is entire, and ϕ and multiplication are continuous. Since $z \mapsto \alpha_z(a)$ is entire, the function $[0, \beta] \to \mathbb{R}$ given by $\gamma \mapsto \|\alpha_{i\gamma}(a)\|$ is continuous, and hence bounded. For $M := \sup\{\|\alpha_{i\gamma}(a)\| : \gamma \in [0, \beta]\}$ we have

$$|f_{a,b}(t + i\gamma)| = |\phi(b\alpha_{t_i\gamma}(a))| \le M\|b\|.$$

So $f_{a,b}$ is bounded on Ω_β. We have $f_{a,b}(t) = \phi(b\alpha_t(a))$ and

$$f_{a,b}(t + i\beta) = \phi(b\alpha_{t+i\beta}(a)) = \phi(b\alpha_{i\beta}(\alpha_t(a))) = \phi(\alpha_t(a)b),$$

where the last equality follows from the fact that a α-analytic implies that $\alpha_t(a)$ α-analytic for all $t \in \mathbb{R}$.

For general $a, b \in A$, we choose certain sequences of α-analytic elements $(a_n), (b_n)$ with $a_n \to a$ and $b_n \to b$. We show that (f_{a_n,b_n}) is Cauchy, and we set $f_{a,b} := \lim_{n\to\infty} f_{a_n,b_n}$.

For $(iii) \implies (i)$, we set for all α-analytic $a, b \in A$,

$$g_{a,b}(z) = \phi(b\alpha_z(a)).$$

Then $g_{a,b}$ is entire, and $g_{a,b}(t) = f_{a,b}(t)$ for all $t \in \mathbb{R}$. Some complex analysis then gives that $g_{a,b}(z) = f_{a,b}(z)$ for all $z \in \Omega_\beta$. Then $\phi(b\alpha_{i\beta}(a)) = g_{a,b}(i\beta) = f_{a,b}(i\beta) = \phi(ab)$.

Proposition 2.10. *Let (A, α) be a C^*-dynamical system, ϕ a state on A, and $0 < \beta < \infty$. If ϕ is KMS$_\beta$, then $\phi = \phi \circ \alpha_t$ for all $t \in \mathbb{R}$.*

3. Directed graph C^*-algebras and their KMS states

For many more details on the following, see Iain's Raeburn's book on graph C^*-algebras [9].

3.1. **Projections and partial isometries.** We start with a brief discussion on projections and partial isometries.

Let M be a closed subspace of a Hilbert space H. The *orthogonal projection* of H onto M is the operator $P \in B(H)$ characterised by the property that $Ph \in M$ and $h - Ph \in M^\perp$ for all $h \in H$. Here, $M^\perp := \{k \in H : (k|h) = 0 \text{ for all } h \in M\}$. It turns out that an operator $P \in B(H)$ is the orthogonal projection onto PH if and only if $P = P^2 = P^*$.

Definition 3.1. An element p in a C^*-algebra A is a *projection* if $p = p^2 = p^*$.

An operator $S \in B(H)$ is an *isometry* if $\|Sh\| = \|h\|$ for all $h \in H$. An operator $S \in B(H)$ is a *partial isometry* if $\|Sh\| = \|h\|$ for all $h \in (\ker S)^\perp$. The following are equivalent:

(1) S is a partial isometry;
(2) S^*S is a projection;
(3) $SS^*S = S$;
(4) SS^* is a projection; and
(5) $S^*SS^* = S^*$.

The projection S^*S is the projection onto $(\ker S)^\perp$, and SS^* the projection onto SH.

Definition 3.2. An element s in a C^*-algebra A is a *partial isometry* if any of (2)–(5) are satisfied. We call it an *isometry* if $s^*s = 1$.

We will look at C^*-algebras generated by projections and partial isometries. We will use the following facts.

Lemma 3.3. *Let A be a C^*-algebra and X a subset of A. Then there is a smallest C^*-subalgebra $C^*(X)$ of A which contains X. If X is closed under multiplication and involution, then*

$$C^*(X) = \overline{\operatorname{span}} X = \overline{\{\sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N},\ \lambda_i \in \mathbb{C}, x_i \in X\}}.$$

Note that we take $C^*(X)$ to be $\bigcap\{B \subseteq A : X \subseteq B,\ B \text{ a } C^*\text{-algebra}\}$.

3.2. **Directed graphs.** A *directed graph* is a tuple $E = (E^0, E^1, r, s)$ where E^0 and E^1 are countable sets, and r, s are functions from E^1 to E^0. We call the elements of E^0 the *vertices* and the elements of E^1 the *edges*. We call r the *range* map, and s the *source* map. We say E is *finite* if E^0 and E^1 are finite. A *path* in E is a sequence of edges $e_1 \dots e_n$ such that $s(e_i) = r(e_{i+1})$ for all $1 \le i \le n-1$. We can extend the range and source maps to paths in the obvious way. We denote by E^* the set of paths of length n (meaning they are paths consisting of n edges), and by $E^* = \bigcup_{n\in\mathbb{N}} E^n$ the set of all finite paths. We say E is *strongly connected* if there is path from any vertex in E to any other vertex in E.

3.3. **Directed graph C^*-algebras.** We will assume throughout that E is a finite graph.

Definition 3.4. A *Toeplitz–Cuntz–Krieger E-family* (TCK E-family) in a C^*-algebra A consists of mutually orthogonal projections $\{P_v : v \in E^0\} \subseteq A$ (so $u \neq v \implies P_uP_v = 0$), and partial isometries $\{S_e : e \in E^1\} \subseteq A$ such that

(TCK1) $S_e^*S_e = P_{s(e)}$ for all $e \in E^1$; and
(TCK2) $P_v \ge \sum_{r(e)=v} S_eS_e^*$ for all $v \in E^0$.

(The sum in (TCK2) is zero when $r^{-1}(v) = \emptyset$.) The family is a *Cuntz–Krieger E-family* (CK E-family) if we have equality in (TCK2). We denote such a Toeplitz–Cuntz–Krieger E-family by $\{P, S\}$.

Example 3.5. Consider $\ell^2(E^*) = \overline{\operatorname{span}}\{\delta_\mu : \mu \in E^*\}$. For $v \in E^0$ let Q_v be the projection onto $\overline{\operatorname{span}}\{\delta_\mu : r(\mu) = v\}$, and for each $e \in E^1$ let

$$T_e(\delta_\mu) = \begin{cases} \delta_{e\mu} & \text{if } s(e) = r(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

You can check that the adjoint is given by

$$T_e^*(\delta_\mu) = \begin{cases} \delta_{\mu'} & \text{if } \mu = e\mu' \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$T_e^*T_e(\delta_\mu) = \begin{cases} T_e^*(\delta_{e\mu}) & \text{if } s(e) = r(\mu) \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \delta_\mu & \text{if } s(e) = r(\mu) \\ 0 & \text{otherwise.} \end{cases},$$

and so $T_e^*T_e = Q_{s(e)}$. Similarly, $T_eT_e^*$ is the projection onto $\overline{\operatorname{span}}\{\delta_{e\mu} : s(e) = r(\mu)\}$, and hence $Q_v \geq \sum_{r(e)=v} T_eT_e^*$. Note that this will be a strict inequality because the subspace $\mathbb{C}\delta_v$ is contained in the range of Q_v but not in the range of $\sum_{r(e)=v} T_eT_e^*$. For a Cuntz–Krieger family E-family we could use the infinite-path space E^∞ (defined in the obvious way) instead of E^*.

For a path $\mu = e_1 \dots e_n$ we write S_μ for the product $S_{e_1} \cdots S_{e_n}$. If $\mu = v \in E^0$, we sometimes write S_v for P_v.

Exercise 3.6. *Show that for $\mu, \nu, \lambda, \eta \in E^*$ with $s(\mu) = s(\nu)$ and $s(\lambda) = s(\eta)$, we have*

$$S_\mu S_\nu^* S_\lambda S_\eta^* = \begin{cases} S_{\mu\lambda'} S_\eta^* & \text{if } \lambda = \nu\lambda', \\ S_\mu S_{\eta\nu'}^* & \text{if } \nu = \lambda\nu', \\ 0 & \text{otherwise.} \end{cases}$$

Note that if we take $\nu = \lambda$, $\mu = s(\nu) = \eta$, then

$$S_\nu^* S_\nu = P_{s(\nu)} S_\nu S_\nu^* P_{s(\nu)}^* = P_{s(\nu)}.$$

Corollary 3.7. *If $\{P, S\}$ is a TCK E-family in a C^*-algebra A, then*

$$C^*(\{P, S\}) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}.$$

Theorem 3.8. *There is a C^*-algebra $\mathcal{T}C^*(E)$ universal for C^*-algebras generated by a TCK E-family, in the sense that $\mathcal{T}C^*(E)$ is generated by a TCK E-family $\{p, s\}$, and whenever $\{P, S\}$ is a TCK E-family in a C^*-algebra A, there is a homomorphism $\pi_{P,S} \colon \mathcal{T}C^*(E) \to A$ satisfying $\pi_{P,S}(p_v) = P_v$ for all $v \in E^0$, and $\pi_{P,S}(s_e) = S_e$ for all $e \in E^1$. We call $\mathcal{T}C^*(E)$ the* Toeplitz–Cuntz–Krieger algebra *(or* Toeplitz algebra*) of E.*

The quotient of $\mathcal{T}C^*(E)$ by the ideal generated by

$$\{p_v - \sum_{r(e)=v} s_e s_e^* : v \in E^0\}$$

is the Cuntz–Krieger algebra *(or* graph algebra*) $C^*(E)$. It is the universal C^*-algebra generated by a CK E-family.*

Note that we have

$$\mathcal{T}C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}.$$

Examples 3.9.
1. The Toeplitz algebra of the graph with one vertex and n loops is the Toeplitz–Cuntz algebra $\mathcal{T}\mathcal{O}_n$, and the graph algbera of this graph is the Cuntz algebra $\mathcal{O}_n$.
2. Recall that TCK E-family $\{Q, T\} \subseteq B(\ell^2(E^*))$. The universal property gives a homomorphism $\pi_{Q,T} \colon \mathcal{T}C^*(E) \to B(\ell^2(E^*))$, and it turns out that this is an isomorphism onto $C^*(\{Q, T\})$.

3.4. **KMS states on the C^*-algebras of finite graphs.** We continue to assume E is finite. All the results in this section are from [5]. We will now introduce a dynamics on $\mathcal{T}C^*(E)$ for which we will study KMS states.

Lemma 3.10. *There is a strongly continuous action $\alpha\colon \mathbb{R} \to \operatorname{Aut} \mathcal{T}C^*(E)$ such that $\alpha_t(p_v) = p_v$ for all $v \in E^0$, and $\alpha_t(s_e) = e^{it}s_e$ for all $e \in E^1$. This action descends to an action α on the graph algebra $C^*(E)$.*

Proof. Fix $z \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. It is easy to see that $\{p_v : v \in E^0\}$ and $\{zs_e : e \in E^1\}$ satisfy (TCK1) and (TCK2). The universal property gives a homomorphism $\gamma_z\colon \mathcal{T}C^*(E) \to \mathcal{T}C^*(E)$ satisfying $\gamma_z(p_v) = p_v$ for all $v \in E^0$, and $\gamma_z(s_e) = zs_e$ for all $e \in E^1$. It is not too hard to prove that we have a strongly continuous action $\gamma\colon \mathbb{T} \to \operatorname{Aut} \mathcal{T}C^*(E)$ (for the continuity of $z \mapsto \gamma_z(a)$, use an $\epsilon/3$-argument). The action γ is called the gauge action. We take $\alpha_t = \gamma_{e^{it}}$. □

We see that $\alpha_t(s_\mu s_\nu^*) = e^{it(|\mu|-|\nu|)}s_\mu s_\nu^*$, and so each $s_\mu s_\nu^*$ is α-analytic. It is enough to check the KMS condition on elements $s_\mu s_\nu^*$.

We start with an algebraic characterisation of a KMS state.

Lemma 3.11 (The algebraic characterisation)**.** *A state ϕ on $\mathcal{T}C^*(E)$ is a KMS$_\beta$ state for α is and only if*

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu}e^{-\beta|\mu|}\phi(p_{s(\mu)}). \tag{3.1}$$

Proof. For the "only if" direction, suppose ϕ is a KMS$_\beta$ state. For $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$ we have

$$\phi(s_\mu s_\nu^*) = \phi(s_\nu^*\alpha_{i\beta}(s_\mu)) = e^{-\beta|\mu|}\phi(s_\nu^* s_\mu) = e^{-\beta|\mu|}\phi(s_\mu\alpha_{i\beta}(s_\nu^*)) = e^{-\beta(|\mu|-|\nu|)}\phi(s_\mu s_\nu^*).$$

So $\phi(s_\mu s_\nu^*) \neq 0 \implies |\mu| = \nu$. But then $s_\nu^* s_\mu \neq 0 \implies \nu = \mu\nu'$ or $\mu = \nu\mu'$, and so we must have $\mu = \nu$. Then

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,nu}e^{-\beta|\mu|}\phi(s_\mu^* s_\mu) = \delta_{\mu,nu}e^{-\beta|\mu|}\phi(p_{s(\mu)}).$$

We leave the "if" direction as an exercise, but you need to show that (3.1) implies that

$$\phi(s_\mu s_\nu^* s_\lambda s_\eta^*) = \phi(s_\lambda s_\eta^*\alpha_{i\beta}(s_\mu s_\nu^*)) = e^{-\beta(|\mu|-|\nu|)}\phi(s_\lambda s_\nu^* s_\mu s_\nu^*).$$

□

Suppose ϕ is a KMS$_\beta$ state on $(\mathcal{T}C^*(E), \alpha)$. We use (TCK2) and the positivity of ϕ to get

$$\phi(p_v) \geq \sum_{r(e)=v} \phi(s_e s_e^*) = \sum_{r(e)=v} e^{-\beta}\phi(p_{s(e)}) \tag{3.2}$$

We define the *vertex matrix* $A_E \in M_{E^0}(\mathbb{N})$ of E by

$$A_E(v, w) = |r^{-1}(v) \cap s^{-1}(w)|.$$

Then we can rewrite (3.2) as

$$e^\beta\phi(p_v) \geq \sum_{w\in E^0} \sum_{\substack{e\in E^1\\ r(e)=v\\ s(e)=w}} \phi(p_w) = \sum_{w\in E^0} A_E(v, w)\phi(p_w).$$

So the vector $m = (m_v) \coloneqq (\phi(p_v)) \in [0, \infty)^{E^0}$ satisfies

$$A_E m \leq e^\beta m.$$

We say that m is a *subinvariant vector* for A_E. If ϕ factors through a KMS$_\beta$ state of $C^*(E)$, then we have equality throughout the above calculations, and then m satisfies $A_E m = e^\beta m$. To analyse these vectors m, we use the following result.

Theorem 3.12 (The Perron–Frobenius Theorem). *Let $A \in M_n([0,\infty))$ be an irreducible matrix. Then*

(1) *the spectral radius $\rho(A)$ is an eigenvalue of A;*
(2) *the eigenspace for $\rho(A)$ is 1-dimensional and contains a positive eigenvector with $\|\cdot\|_1$-norm equal to 1; and*
(3) *if $y \in [0,\infty)^n \setminus \{0\}$ and $\lambda \in \mathbb{R}$, then $Ay \leq \lambda y \iff \lambda \geq \rho(A)$.*

So assume E is strongly connected, and ϕ is a KMS_β state that factors through $C^*(E)$. Then the Perron–Frobenius Theorem says that e^β is the spectral radius $\rho(A_E)$, and since

$$1 = \phi(1) = \sum_{v \in E^0} \phi(p_v) = \sum_{v \in E^0} m_v = \|m\|_1,$$

m is the unique eigenvector in the eigenspace of $\rho(A_E)$ with $\|m\|_1 = 1$. We see from the algebraic characterisation that m completely determines the state ϕ. So we get the following theorem of Enomoto–Fujii–Watatana [3]:

Theorem 3.13. *Suppose E is a strongly connected directed graph. Then $(C^*(E),\alpha)$ has at most one KMS_β state, and this state has inverse temperature* $\log\rho(A_E)$.

Note that such $C^*(E)$ are Cuntz–Krieger algebras, and this result generalises the Olesen–Pedersen result [7] that says that $(\mathcal{O}_n,\alpha)$ has a single KMS state, and the inverse temperature is $\log n$.

We now just assume that E is finite. Take $\beta > \log\rho(A_E)$, $m = (\phi(p_v))$, and $\varepsilon := (1 - e^{-\beta}A_E)m$. Then $\varepsilon \geq 0$ and is nonzero. Since e^β is not in the spectrum of A_E, we have $m = (1 - e^{-\beta}A_E)^{-1}\varepsilon$.

Exercise 3.14. *The series $\sum_{n=0}^\infty e^{-\beta n}A_E^n$ converges in operator norm to $(1-e^{-\beta}A_E)^{-1}$. (See* [2, VII.3.4]*.)*

Since $A_E(v,w)$ is the number of paths of length n from w to v, we have

$$\sum_{s(\mu)=w} e^{-\beta|\mu|} = \sum_{n=0}^\infty \sum_{\substack{\mu \in E^n \\ s(\mu)=w}} e^{-\beta n} = \sum_{n=0}^\infty \sum_{v\in E^0} e^{-\beta n} A_E^n(v,w),$$

which converges because of the exercise. We set

$$y^\beta = (y_v^\beta) := \Big(\sum_{s(\mu)=v} e^{-\beta|\mu|}\Big) \in [0,\infty)^{E^0}.$$

Then you can show that $\|m\|_1 = \varepsilon \cdot y^\beta$. We now have:

Theorem 3.15 (an Huef–Laca–Raeburn–Sims). *Suppose E is a finite graph, and $\beta > \log\rho(A_E)$. Suupose $\varepsilon \cdot y^\beta = 1$. Then there is a KMS_β state ϕ_ε of $\mathcal{T}C^*(E)$ such that*

$$\phi_\varepsilon(p_v) = ((1-e^{-\beta}A_E)^{-1}\varepsilon)_v$$

for all $v \in E^0$. The map $\varepsilon \mapsto \phi_\varepsilon$ is an affine isomorphism of $\{\varepsilon : \varepsilon \cdot y^\beta = 1\}$ onto the simplex of KMS_β states of $(\mathcal{T}C^*(E),\alpha)$.

Remark 3.16.

(1) an Huef–Laca–Raeburn–Sims use a spatial argument using the representation of $\mathcal{T}C^*(E)$ on $\ell^2(E^*)$ to get ϕ_ε.
(2) The above analysis does not apply to $\beta = \log\rho(A_E)$ because $1 - e^{-\beta}A_E$ is not invertible. But there is a $\mathrm{KMS}_{\log\rho(A_E)}$ state on $(\mathcal{T}C^*(E),\alpha)$ because you can take β_n converging to $\log\rho(A_E)$, and then compactness of the state space gives you a sequence of KMS_{β_n} states converging weak-$*$ to a $\mathrm{KMS}_{\log\rho(A_E)}$ state.

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